

Logarithmic tensor product theory for generalized modules for a conformal vertex algebra, Part I

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Abstract

We generalize the tensor product theory for modules for a vertex operator algebra previously developed in a series of papers by the first two authors to suitable module categories for a “conformal vertex algebra” or even more generally, for a “Möbius vertex algebra.” We do not require the module categories to be semisimple, and we accommodate modules with generalized weight spaces. As in the earlier series of papers, our tensor product functors depend on a complex variable, but in the present generality, the logarithm of the complex variable is involved. This first part is devoted to the study of logarithmic intertwining operators and their role in the construction of the tensor product functors. Part II of this work will be devoted to the construction of the appropriate natural associativity isomorphisms between triple tensor product functors, to the proof of their fundamental properties, and to the construction of the resulting braided tensor category structure. This work includes the complete proofs in the present generality and can be read independently of the earlier series of papers.

Part I

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1 Introduction

In a series of papers ([HL1], [HL4], [HL5], [HL6], [HL7], [H1]), the first two authors have developed a tensor product theory for modules for a vertex operator algebra under suitable conditions. A structure called “vertex tensor category structure,” which is much richer than tensor category structure, has thereby been established for many important categories of modules for classes of vertex operator algebras, since the conditions needed for invoking the general theory have been verified for these categories. The most important such families of examples of this theory are listed below. In the present work, which has been announced in [HLZ], we generalize this tensor product theory to a larger family of module categories, for a “conformal vertex algebra,” or even more generally, for a “Möbius vertex algebra,” under suitably relaxed conditions. A conformal vertex algebra is just a vertex algebra in the sense of Borchers [B] equipped with a conformal vector satisfying the usual axioms; a Möbius vertex algebra is a variant of a “quasi-vertex operator algebra” as in [FHL]. A central feature of the present work is that we do not require the module categories to be semisimple and that we accommodate modules with generalized weight spaces.

As in the earlier series of papers, our tensor product functors depend on a complex variable, but in the present generality, the logarithm of the complex variable is involved. The first part of this work is devoted to the study of logarithmic intertwining operators and their role in the construction of the tensor product functors. The remainder of this work is devoted to the construction of the appropriate natural associativity isomorphisms between triple tensor product functors, to the proof of their fundamental properties, and to the construction of the resulting braided tensor category structure. This leads to vertex tensor category structure for further important families of examples, or, in the Möbius case, to “Möbius vertex tensor category” structure.

This work includes the complete proofs in the present generality and can be read independently of the earlier series of papers. Our treatment is based on the theory of vertex operator algebras and their modules, as developed in [FLM2], [FHL], [DL] and [LL].

The main families for which the conditions needed for invoking the first two authors’ general tensor product theory have been verified, thus yielding vertex tensor category structure on these module categories, are the module categories for the following classes of vertex operator algebras (or, in the last case, vertex operator superalgebras):

1. The vertex operator algebras V_L associated with positive definite even lattices L ; see [B], [FLM2] for these vertex operator algebras and see [D1], [DL] for the conditions needed for invoking the general tensor product theory.
2. The vertex operator algebras $L(k, 0)$ associated with affine Lie algebras and positive integers k ; see [FZ] for these vertex operator algebras and [FZ], [HL8] for the conditions.
3. The “minimal series” of vertex operator algebras associated with the Virasoro algebra; see [FZ] for these vertex operator algebras and [W], [H2] for the conditions.
4. Frenkel, Lepowsky and Meurman’s moonshine module V^\natural ; see [FLM1], [B], [FLM2] for this vertex operator algebra and [D2] for the conditions.

5. The fixed point vertex operator subalgebra of V^{\natural} under the standard involution; see [FLM1], [FLM2] for this vertex operator algebra and [D2], [H3] for the conditions.
6. The “minimal series” of vertex operator superalgebras (suitably generalized vertex operator algebras) associated with the Neveu-Schwarz superalgebra and also the “unitary series” of vertex operator superalgebras associated with the $N = 2$ superconformal algebra; see [KW] and [A2] for the corresponding $N = 1$ and $N = 2$ vertex operator superalgebras, respectively, and [A1], [A3], [HM1], [HM2] for the conditions.

In addition, vertex tensor category structure has also been established for the module categories for certain vertex operator algebras built from the vertex operator algebras just mentioned, such as tensor products of such algebras; this is carried out in certain of the papers listed above.

For all of the six classes of vertex operator algebras (or superalgebras) listed above, each of the algebras is “rational” in the specific sense of Huang-Lepowsky’s work on tensor product theory. This particular “rationality” property is easily proved to be a sufficient condition for insuring that the tensor product modules exist; see for instance [HL5]. Unfortunately, the phrase “rational vertex operator algebra” also has several other distinct meanings in the literature. Thus we find it convenient at this time to assign a new term, “finite reductivity,” to our particular notion of “rationality”: We say that a vertex operator algebra (or superalgebra) V is *finitely reductive* if

1. Every V -module is completely reducible.
2. There are only finitely many irreducible V -modules (up to equivalence).
3. All the fusion rules (the dimensions of the spaces of intertwining operators among triples of modules) for V are finite.

We choose the term “finitely reductive” because we think of the term “reductive” as describing the complete reducibility—the first of the conditions (that is, the algebra “(completely) reduces” every module); the other two conditions are finiteness conditions.

The vertex-algebraic study of tensor category structure on module categories for certain vertex algebras was stimulated by the work of Moore and Seiberg [MS], in which, in the study of what they termed “rational” conformal field theory, they constructed a tensor category structure based on the assumption of the existence of a suitable operator product expansion for “chiral vertex operators” (which correspond to intertwining operators in vertex algebra theory). Earlier, in [BPZ], Belavin, Polyakov, and Zamolodchikov had already formalized the relation between the (nonmeromorphic) operator product expansion, chiral correlation functions and representation theory, for the Virasoro algebra in particular, and Knizhnik and Zamolodchikov [KZ] had established fundamental relations between conformal field theory and the representation theory of affine Lie algebras. As we have discussed in the introductory material in [HL4], [HL5] and [HL8], such study of conformal field theory is deeply connected with the vertex-algebraic construction and study of tensor categories, and also with other mathematical approaches to the construction of tensor categories in the

spirit of conformal field theory. Concerning the latter approaches, we would like to mention that the method used by Kazhdan and Lusztig, especially in their construction of the associativity isomorphisms, in their breakthrough work in [KL1]–[KL5], is related to the algebro-geometric formulation and study of conformal-field-theoretic structures in the influential works of Tsuchiya-Ueno-Yamada [TUY], Drinfeld [Dr] and Beilinson-Feigin-Mazur [BFM]. See also the important work of Deligne [De], Finkelberg ([Fi1] [Fi2]), Bakalov-Kirillov [BK] and Nagatomo-Tsuchiya [NT] on the construction of tensor categories in the spirit of this approach to conformal field theory.

The semisimplicity of the module categories mentioned in the examples above is related to another property of these modules, namely, that each module is a direct sum of its “weight spaces,” which are the eigenspaces of a special operator $L(0)$ coming from the Virasoro algebra action on the module. But there are important situations in which module categories are not semisimple and in which modules are not direct sums of their weight spaces. Notably, for the vertex operator algebras $L(k, 0)$ associated with affine Lie algebras, when the sum of k and the dual Coxeter number of the corresponding Lie algebra is not a nonnegative rational number, the vertex operator algebra $L(k, 0)$ is not finitely reductive, and, working with Lie algebra theory rather than with vertex operator algebra theory, Kazhdan and Lusztig constructed a natural braided tensor category structure on a certain category of modules of level k for the affine Lie algebra ([KL1], [KL2], [KL3], [KL4], [KL5]). This work of Kazhdan-Lusztig in fact motivated the first two authors to develop an analogous theory for vertex operator algebras rather than for affine Lie algebras, as was explained in detail in the introductory material in [HL1], [HL4], [HL5], [HL6], and [HL8]. However, this general theory, in its original form, does not apply to Kazhdan-Lusztig’s context, because the vertex-operator-algebra modules considered in [HL1], [HL4], [HL5], [HL6], [HL7], [H1] are assumed to be the direct sums of their weight spaces (with respect to $L(0)$), and the non-semisimple modules considered by Kazhdan-Lusztig fail in general to be the direct sums of their weight spaces. Although their setup, based on Lie theory, and ours, based on vertex operator algebra theory, are very different (as was discussed in the introductory material in our earlier papers), we expected to be able to recover (and further extend) their results through our vertex operator algebraic approach, which is very general, as we discussed above. This motivated us, in the present work, to generalize the work of the first two authors by considering modules with generalized weight spaces, and especially, intertwining operators associated with such generalized kinds of modules. This allows us to construct braided tensor category structure, and even vertex tensor category structure, on important module categories that are not semisimple. Using the present theory, the third author ([Z1], [Z2]) has indeed recovered the braided tensor category structure of Kazhdan-Lusztig, and also extended it to vertex tensor category structure.

Logarithmic structure in conformal field theory was in fact first introduced by physicists to describe disorder phenomena [Gu]. A lot of progress has been made on this subject. We refer the interested reader to the review articles [Ga2], [Fl2], [RT] and [Fu], and references therein, in particular, for example, [CF], [FGST1], [FGST2], [FHST], [Fl1], [Ga1], [GK1] and [GK2]. The paper [FHST] initiates an interesting direction in logarithmic conformal field

theory. The paper [CF] in fact uses the results in the present work as announced in [HLZ].

Such logarithmic structures also arise naturally in the representation theory of vertex operator algebras. In fact, in the construction of intertwining operator algebras, the first author proved (see [H6]) that if modules for the vertex operator algebra satisfy a certain cofiniteness condition, then products of the usual intertwining operators satisfy certain systems of differential equations with regular singular points. In addition, it was proved in [H6] that if the vertex operator algebra satisfies certain finite reductivity conditions, then the analytic extensions of products of the usual intertwining operators have no logarithmic terms. In the case when the vertex operator algebra satisfies only the cofiniteness condition but not the finite reductivity conditions, the products of intertwining operators still satisfy systems of differential equations with regular singular points. But in this case, the analytic extensions of such products of intertwining operators might have logarithmic terms. This means that if we want to generalize the results in [HL1], [HL4]–[HL7], [H1] and [H6] to the case in which the finite reductivity properties are not always satisfied, we have to consider intertwining operators involving logarithmic terms.

In [Mi], Milas introduced and studied what he called “logarithmic modules” and “logarithmic intertwining operators.” Roughly speaking, logarithmic modules are weak modules for a vertex operator algebra that are direct sums of generalized eigenspaces for the operator $L(0)$. We will call such weak modules “generalized modules” in this work. Logarithmic intertwining operators are operators that depend not only on powers of a (formal or complex) variable x , but also on its logarithm $\log x$.

The special features of the logarithm function make the logarithmic theory very subtle and interesting. Although we show that all the main theorems in the original tensor product theory developed by the first two authors still hold in the logarithmic theory, many of the proofs involve certain new techniques and have surprising connections with certain combinatorial identities.

As we mentioned above, one important application of our generalization is to the category \mathcal{O}_κ of certain modules for an affine Lie algebra studied by Kazhdan and Lusztig in their series of papers [KL1]–[KL5]. It has been shown in [Z1] and [Z2] by the third author that, among other things, all the conditions needed to apply our theory to this module category are satisfied. As a result, it is proved in [Z1] and [Z2] that there is a natural vertex tensor category structure on this module category, giving in particular a new construction, in the context of general vertex operator algebra theory, of the braided tensor category structure on \mathcal{O}_κ . The methods used in [KL1]–[KL5] were very different.

In addition to these logarithmic issues, another way in which the present work generalizes the earlier tensor product theory for module categories for a vertex operator algebra is that we now allow the algebras to be somewhat more general than vertex operator algebras, in order, for example, to accommodate module categories for the vertex algebras V_L where L is a nondegenerate even lattice that is not necessarily positive definite (cf. [B], [DL]); see [Z1].

What we accomplish in this work, then, is the following: We generalize essentially all the results in [HL5], [HL6], [HL7] and [H1] from the category of modules for a vertex operator algebra to categories of suitably generalized modules for a conformal vertex algebra or a

Möbius vertex algebra equipped with an additional suitable grading by an abelian group. The algebras that we consider include not only vertex operator algebras but also such vertex algebras as V_L where L is a nondegenerate even lattice, and the modules that we consider are not required to be the direct sums of their weight spaces but instead are required only to be the (direct) sums of their “generalized weight spaces,” in a suitable sense. In particular, in this work we carry out, in the present greater generality, the construction theory for the “ $P(z)$ -tensor product” functor originally done in [HL5], [HL6] and [HL7] and the associativity theory for this functor—the construction of the natural associativity isomorphisms between suitable “triple tensor products” and the proof of their important properties, including the isomorphism property—originally done in [H1]. This leads, as in [HL9], to the proof of the coherence properties for vertex tensor categories, and in the Möbius case, the coherence properties for Möbius vertex tensor categories.

The general structure of much of this work essentially follows that of [HL5], [HL6], [HL7] and [H1]. However, the results here are much stronger and more general than in these earlier works, and in addition, many of the results here have no counterparts in those works. Moreover, many ideas, formulations and proofs in this work are necessarily quite different from those in the earlier papers, and we have chosen to give some proofs that are new even in the finitely reductive case studied in the earlier papers.

Some of the new ingredients that we are introducing into the theory here are (as we shall explain in detail): an analysis of logarithmic intertwining operators, including “logarithmic formal calculus”; a notion of “ $P(z_1, z_2)$ -intertwining map” and a study of its properties; new “compatibility conditions”; a generalization of the result that the homogeneous components of the products and iterates of intertwining maps span the appropriate tensor product modules; results strengthening the relation between products and iterates of intertwining maps; and a generalized sufficient condition for the applicability of the theory, a condition that can be applied in the case of our suitably generalized modules.

The contents of the sections of this work are as follows: In the rest of this Introduction we compare classical tensor product theory for Lie algebra modules with tensor product theory for vertex operator algebra modules. A crucial difference between the two theories is that in the vertex operator algebra setting, the theory depends on an “extra parameter” z , which must be understood as a (nonzero) complex variable rather than as a formal variable (although one needs, and indeed we very heavily use, an extensive “calculus of formal variables” in order to develop the theory). In Section 2 we recall some basic concepts in the theory of vertex (operator) algebras. We use the treatments of [FLM2], [FHL], [DL] and [LL]; in particular, it is crucial in this tensor product theory to use the formal-calculus point of view as developed in these works. Readers can consult these works for further detail. We also set up the notation that will be used in this work, and we describe the main category of (generalized) modules that we will consider. In Section 3 we introduce the notion of logarithmic intertwining operator as in [Mi] and present a detailed study of some of its properties. In Section 4 and 5 we present the definitions and constructions of $P(z)$ - and $Q(z)$ -tensor products, generalizing those in [HL5], [HL6] and [HL7]. Some of the proofs of results in Section 5 are given in Section 6. In Section 7 the convergence condition introduced

in [H1] for constructing the associativity isomorphism is given in the present context. The new notion of $P(z_1, z_2)$ -intertwining map, generalizing the corresponding concept in [H1], is introduced in Section 8. This will play a crucial role in the construction of the associativity isomorphisms. In Section 9 we prove some properties that are satisfied by vectors in the dual space of the vector-space tensor product of three modules that arise from products and from iterates of intertwining maps. This enables us to define two crucial subspaces of this dual space, by means of suitable compatibility and local grading restriction conditions. By relating these two subspaces, we construct the associativity isomorphism in Section 10. In Section 11, we generalize a certain sufficient condition for the existence of associativity isomorphisms in [H1], and we prove the relevant conditions using differential equations. In Section 12, we establish the coherence of our tensor category.

1.1 The Lie algebra case

It is heuristically useful to start by considering the tensor product theory for modules for a Lie algebra in a somewhat unusual way—a way that motivates our approach for the case of vertex algebras.

In the theory of tensor products for modules for a Lie algebra, the tensor product of two modules is defined, or rather, constructed, as the vector-space tensor product of the two modules, equipped with a Lie algebra module action given by the familiar diagonal action of the Lie algebra. In the vertex algebra case, however, the vector-space tensor product of two modules for a vertex algebra is *not* the correct underlying vector space for the tensor product of the vertex-algebra modules. In this subsection we therefore consider another approach to the tensor product theory for modules for a Lie algebra—an approach, based on “intertwining maps,” that will show how the theory proceeds in the vertex algebra case. Then, in the next subsection, we shall lay out the corresponding “road map” for the tensor product theory in the vertex algebra case, which we then carry out in the body of this work.

We first recall the following elementary but crucial background about tensor product vector spaces: Given vector spaces W_1 and W_2 , the corresponding tensor product structure consists of a vector space $W_1 \otimes W_2$ equipped with a bilinear map

$$W_1 \times W_2 \longrightarrow W_1 \otimes W_2,$$

denoted

$$(w_{(1)}, w_{(2)}) \mapsto w_{(1)} \otimes w_{(2)}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that for any vector space W_3 and any bilinear map

$$B : W_1 \times W_2 \longrightarrow W_3,$$

there is a unique linear map

$$L : W_1 \otimes W_2 \longrightarrow W_3$$

such that

$$B(w_{(1)}, w_{(2)}) = L(w_{(1)} \otimes w_{(2)})$$

for $w_{(i)} \in W_i$, $i = 1, 2$. This universal property characterizes the tensor product structure $W_1 \otimes W_2$, equipped with its bilinear map $\cdot \otimes \cdot$, up to unique isomorphism. In addition, the tensor product structure in fact exists.

As was illustrated in [HL4], and as is well known, the notion of tensor product of modules for a Lie algebra can be formulated in terms of what can be called “intertwining maps”: Let W_1, W_2, W_3 be modules for a fixed Lie algebra V . (We are calling our Lie algebra V because we shall be calling our vertex algebra V , and we would like to emphasize the analogies between the two theories.) An *intertwining map of type* $\binom{W_3}{W_1 W_2}$ is a linear map $I : W_1 \otimes W_2 \longrightarrow W_3$ (or equivalently, from what we have just recalled, a bilinear map $W_1 \times W_2 \longrightarrow W_3$) such that

$$\pi_3(v)I(w_{(1)} \otimes w_{(2)}) = I(\pi_1(v)w_{(1)} \otimes w_{(2)}) + I(w_{(1)} \otimes \pi_2(v)w_{(2)}) \quad (1.1)$$

for $v \in V$ and $w_{(i)} \in W_i$, $i = 1, 2$, where π_1, π_2, π_3 are the module actions of V on W_1, W_2 and W_3 , respectively. (Clearly, such an intertwining map is the same as a module map from $W_1 \otimes W_2$, equipped with the tensor product module structure, to W_3 , but we are now temporarily “forgetting” what the tensor product module is.)

A *tensor product of the V -modules W_1 and W_2* is then a pair (W_0, I_0) , where W_0 is a V -module and I_0 is an intertwining map of type $\binom{W_0}{W_1 W_2}$ (which, again, could be viewed as a suitable bilinear map $W_1 \times W_2 \longrightarrow W_0$), such that for any pair (W, I) with W a V -module and I an intertwining map of type $\binom{W}{W_1 W_2}$, there is a unique module homomorphism $\eta : W_0 \longrightarrow W$ such that $I = \eta \circ I_0$. This universal property of course characterizes (W_0, I_0) up to canonical isomorphism. Moreover, it is obvious that the tensor product in fact exists, and may be constructed as the vector-space tensor product $W_1 \otimes W_2$ equipped with the diagonal action of the Lie algebra, together with the identity map from $W_1 \otimes W_2$ to itself (or equivalently, the canonical bilinear map $W_1 \times W_2 \longrightarrow W_1 \otimes W_2$). We shall denote the tensor product (W_0, I_0) of W_1 and W_2 by $(W_1 \boxtimes W_2, \boxtimes)$, where it is understood that the image of $w_{(1)} \otimes w_{(2)}$ under our canonical intertwining map \boxtimes is $w_{(1)} \boxtimes w_{(2)}$. Thus $W_1 \boxtimes W_2 = W_1 \otimes W_2$, and under our identifications, $\boxtimes = 1_{W_1 \otimes W_2}$.

Remark 1.1 This classical explicit construction of course shows that the tensor product functor exists for the category of modules for a Lie algebra. For vertex algebras, it will be relatively straightforward to *define* the appropriate tensor product functor(s) (see [HL4], [HL5], [HL6], [HL7]), but it will be a nontrivial matter to *construct* this functor (or more precisely, these functors) and thereby prove that the (appropriate) tensor product of modules for a (suitable) vertex algebra exists. The reason why we have formulated the notion of tensor product module for a Lie algebra in the way that we just did is that this formulation motivates the correct notion of tensor product functor(s) in the vertex algebra case.

Remark 1.2 Using this explicit construction of the tensor product functor and our notation $w_{(1)} \boxtimes w_{(2)}$ for the tensor product of elements, the standard natural associativity isomorphisms among tensor products of triples of Lie algebra modules are expressed as follows: Since $w_{(1)} \boxtimes w_{(2)} = w_{(1)} \otimes w_{(2)}$, we have

$$\begin{aligned} (w_{(1)} \boxtimes w_{(2)}) \boxtimes w_{(3)} &= (w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}, \\ w_{(1)} \boxtimes (w_{(2)} \boxtimes w_{(3)}) &= w_{(1)} \otimes (w_{(2)} \otimes w_{(3)}) \end{aligned}$$

for $w_{(i)} \in W_i$, $i = 1, 2, 3$, and so the canonical identification between $w_{(1)} \otimes (w_{(2)} \otimes w_{(3)})$ and $(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}$ gives the standard natural isomorphism

$$\begin{aligned} (W_1 \boxtimes W_2) \boxtimes W_3 &\rightarrow W_1 \boxtimes (W_2 \boxtimes W_3) \\ (w_{(1)} \boxtimes w_{(2)}) \boxtimes w_{(3)} &\mapsto w_{(1)} \boxtimes (w_{(2)} \boxtimes w_{(3)}). \end{aligned} \tag{1.2}$$

This collection of natural associativity isomorphisms of course satisfies the classical coherence conditions for associativity isomorphisms among multiple nested tensor product modules—the conditions that say that in nested tensor products involving any number of tensor factors, the placement of parentheses (as in (1.2), the case of three tensor factors) is immaterial; we shall discuss coherence conditions in detail later. Now, as was discovered in [H1], it turns out that maps analogous to (1.2) can also be constructed in the vertex algebra case, giving natural associativity isomorphisms among triples of modules for a (suitable) vertex operator algebra. However, in the vertex algebra case, the elements “ $w_{(1)} \boxtimes w_{(2)}$,” which indeed exist (under suitable conditions) and are constructed in the theory, lie in a suitable “completion” of the tensor product module rather than in the module itself. Correspondingly, it is a nontrivial matter to construct the triple-tensor-product elements on the two sides of (1.2); in fact, one needs to prove certain convergence, under suitable additional conditions. Even after the triple-tensor-product elements are constructed (in suitable completions of the triple-tensor-product modules), it is a delicate matter to construct the appropriate natural associativity maps, analogous to (1.2), to prove that they are well defined, and to prove that they are isomorphisms. In the present work, we shall generalize these matters (in a self-contained way) from the context of [H1] to a more general one. In the rest of this subsection, for triples of modules for a Lie algebra, we shall now describe a construction of the natural associativity isomorphisms that will seem roundabout and indirect, but this is the method of construction of these isomorphisms that will give us the correct “road map” for the corresponding construction (and theorems) in the vertex algebra case, as in [HL5], [HL6], [HL7] and [H1].

A significant feature of the constructions in the earlier works (and in the present work) is that the tensor product of modules W_1 and W_2 for a vertex operator algebra V is the contragredient module of a certain V -module that is typically a *proper* subspace of $(W_1 \otimes W_2)^*$, the dual space of the vector-space tensor product of W_1 and W_2 . In particular, our treatment, which follows, of the Lie algebra case will use contragredient modules, and we will therefore restrict our attention to *finite-dimensional* modules for our Lie algebra. It will be important that the double-contragredient module of a Lie algebra module is naturally isomorphic to the original module. We shall sometimes denote the contragredient module of a V -module W by W' , so that $W'' = W$. (We recall that for a module W for a Lie algebra V , the corresponding contragredient module W' consists of the dual vector space W^* equipped with the action of V given by: $(v \cdot w^*)(w) = -w^*(v \cdot w)$ for $v \in V$, $w^* \in W^*$, $w \in W$.)

Let us, then, now restrict our attention to finite-dimensional modules for our Lie algebra V . The dual space $(W_1 \otimes W_2)^*$ carries the structure of the classical contragredient module of the tensor product module. Given any intertwining map of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$, using the natural

linear isomorphism

$$\mathrm{Hom}(W_1 \otimes W_2, W_3) \xrightarrow{\sim} \mathrm{Hom}(W_3^*, (W_1 \otimes W_2)^*) \quad (1.3)$$

we have a corresponding linear map in $\mathrm{Hom}(W_3^*, (W_1 \otimes W_2)^*)$, and this must be a map of V -modules. In the vertex algebra case, given V -modules W_1 and W_2 , it turns out that with a suitable analogous setup, the union in the vector space $(W_1 \otimes W_2)^*$ of the ranges of all such V -module maps, as W_3 and the intertwining map vary (and with W_3^* replaced by the contragredient module W_3'), is the correct candidate for the contragredient module of the tensor product module $W_1 \boxtimes W_2$. Of course, in the Lie algebra situation, this union is $(W_1 \otimes W_2)^*$ itself (since we are allowed to take $W_3 = W_1 \otimes W_2$ and the intertwining map to be the canonical map), but in the vertex algebra case, this union is typically much smaller than $(W_1 \otimes W_2)^*$. In the vertex algebra case, we will use the notation $W_1 \boxtimes W_2$ to designate this union, and if the tensor product module $W_1 \boxtimes W_2$ in fact exists, then

$$W_1 \boxtimes W_2 = (W_1 \boxtimes W_2)', \quad (1.4)$$

$$W_1 \boxtimes W_2 = (W_1 \boxtimes W_2)'. \quad (1.5)$$

Thus in the Lie algebra case we will write

$$W_1 \boxtimes W_2 = (W_1 \otimes W_2)^*, \quad (1.6)$$

and (1.4) and (1.5) hold. (In the Lie algebra case we prefer to write $(W_1 \otimes W_2)^*$ rather than $(W_1 \otimes W_2)'$, because in the vertex algebra case, $W_1 \otimes W_2$ is typically not a V -module, and so we will not be allowed to write $(W_1 \otimes W_2)'$ in the vertex algebra case.)

The subspace $W_1 \boxtimes W_2$ of $(W_1 \otimes W_2)^*$ was in fact further described in the following terms in [HL5] and [HL7], in the vertex algebra case: For any map in $\mathrm{Hom}(W_3', (W_1 \otimes W_2)^*)$ corresponding to an intertwining map according to (1.3), the image of any $w'_{(3)} \in W_3'$ under this map satisfies certain subtle conditions, called the “compatibility condition” and the “local grading restriction condition”; these conditions are not “visible” in the Lie algebra case. These conditions in fact precisely describe the proper subspace $W_1 \boxtimes W_2$ of $(W_1 \otimes W_2)^*$. We will discuss such conditions further in Section 1.2 and in the body of this work. As we shall explain, this idea of describing elements in certain dual spaces was also used in constructing the natural associativity isomorphisms between triples of modules for a vertex operator algebra in [H1].

In order to give the reader a guide to the vertex algebra case, we now describe the analogue for the Lie algebra case of this construction of the associativity isomorphisms. To construct the associativity isomorphism from $(W_1 \boxtimes W_2) \boxtimes W_3$ to $W_1 \boxtimes (W_2 \boxtimes W_3)$, it is equivalent (by duality) to give a suitable isomorphism from $W_1 \boxtimes (W_2 \boxtimes W_3)$ to $(W_1 \boxtimes W_2) \boxtimes W_3$ (recall (1.4), (1.5)).

Rather than directly constructing an isomorphism between these two V -modules, it turns out that we want to embed both of them, separately, into the single space $(W_1 \otimes W_2 \otimes W_3)^*$. Note that $(W_1 \otimes W_2 \otimes W_3)^*$ is naturally a V -module, via the contragredient of the diagonal

action, that is,

$$\begin{aligned}
(\pi(v)\lambda)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) &= -\lambda(\pi_1(v)w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
&- \lambda(w_{(1)} \otimes \pi_2(v)w_{(2)} \otimes w_{(3)}) \\
&- \lambda(w_{(1)} \otimes w_{(2)} \otimes \pi_3(v)w_{(3)}),
\end{aligned} \tag{1.7}$$

for $v \in V$ and $w_{(i)} \in W_i$, $i = 1, 2, 3$, where π_1, π_2, π_3 are the module actions of V on W_1 , W_2 and W_3 , respectively. A concept related to this is the notion of *intertwining map from $W_1 \otimes W_2 \otimes W_3$ to a module W_4* , a natural analogue of (1.1), defined to be a linear map

$$F : W_1 \otimes W_2 \otimes W_3 \longrightarrow W_4 \tag{1.8}$$

such that

$$\begin{aligned}
\pi_4(v)F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) &= F(\pi_1(v)w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
&+ F(w_{(1)} \otimes \pi_2(v)w_{(2)} \otimes w_{(3)}) \\
&+ F(w_{(1)} \otimes w_{(2)} \otimes \pi_3(v)w_{(3)}),
\end{aligned} \tag{1.9}$$

with the obvious notation. The relation between (1.7) and (1.9) comes directly from the natural linear isomorphism

$$\text{Hom}(W_1 \otimes W_2 \otimes W_3, W_4) \xrightarrow{\sim} \text{Hom}(W_4^*, (W_1 \otimes W_2 \otimes W_3)^*); \tag{1.10}$$

given F , we have

$$\begin{aligned}
W_4^* &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\
\nu &\mapsto \nu \circ F.
\end{aligned} \tag{1.11}$$

Under this natural linear isomorphism, the intertwining maps correspond precisely to the V -module maps from W_4^* to $(W_1 \otimes W_2 \otimes W_3)^*$. In the situation for vertex algebras, as was the case for tensor products of two rather than three modules, there are analogues of all of the notions and comments discussed in this paragraph *except that we will not put V -module structure onto the vector space $W_1 \otimes W_2 \otimes W_3$* ; as we have emphasized, we will instead base the theory on intertwining maps.

Two important ways of constructing maps of the type (1.8) are as follows: For modules W_1, W_2, W_3, W_4, M_1 and intertwining maps I_1 and I_2 of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, by definition the composition $I_1 \circ (1_{W_1} \otimes I_2)$ is an intertwining map from $W_1 \otimes W_2 \otimes W_3$ to W_4 . Similarly, for intertwining maps I^1, I^2 of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, the composition $I^1 \circ (I^2 \otimes 1_{W_3})$ is an intertwining map from $W_1 \otimes W_2 \otimes W_3$ to W_4 . Hence we have two V -module homomorphisms

$$\begin{aligned}
W_4^* &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\
\nu &\mapsto \nu \circ F_1,
\end{aligned} \tag{1.12}$$

where F_1 is the intertwining map $I_1 \circ (1_{W_1} \otimes I_2)$; and

$$\begin{aligned} W_4^* &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto \nu \circ F_2, \end{aligned} \tag{1.13}$$

where F_2 is the intertwining map $I^1 \circ (I^2 \circ 1_{W_3})$.

The special cases in which the modules W_4 are two iterated tensor product modules and the “intermediate” modules M_1 and M_2 are two tensor product modules are particularly interesting: When $W_4 = W_1 \boxtimes (W_2 \boxtimes W_3)$ and $M_1 = W_2 \boxtimes W_3$, and I_1 and I_2 are the corresponding canonical intertwining maps, (1.12) gives the natural V -module homomorphism

$$\begin{aligned} W_1 \boxtimes (W_2 \boxtimes W_3) &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \nu(w_{(1)} \boxtimes (w_{(2)} \boxtimes w_{(3)}))); \end{aligned} \tag{1.14}$$

when $W_4 = (W_1 \boxtimes W_2) \boxtimes W_3$ and $M_2 = W_1 \boxtimes W_2$, and I^1 and I^2 are the corresponding canonical intertwining maps, (1.13) gives the natural V -module homomorphism

$$\begin{aligned} (W_1 \boxtimes W_2) \boxtimes W_3 &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \nu((w_{(1)} \boxtimes w_{(2)}) \boxtimes w_{(3)})). \end{aligned} \tag{1.15}$$

Clearly, in our Lie algebra case, both of the maps (1.14) and (1.15) are isomorphisms, since they both in fact amount to the identity map on $(W_1 \otimes W_2 \otimes W_3)^*$. However, in the vertex algebra case the analogues of these two maps are only injective homomorphisms, and typically not isomorphisms. (Recall the analogous situation, mentioned above, for double rather than triple tensor products.) These two maps enable us to identify both $W_1 \boxtimes (W_2 \boxtimes W_3)$ and $(W_1 \boxtimes W_2) \boxtimes W_3$ with subspaces of $(W_1 \otimes W_2 \otimes W_3)^*$. In the vertex algebra case we will have certain “compatibility conditions” and “local grading restriction conditions” on elements of $(W_1 \otimes W_2 \otimes W_3)^*$ to describe each of the two subspaces. In either the Lie algebra or the vertex algebra case, the construction of our desired natural associativity isomorphism between the two modules $(W_1 \boxtimes W_2) \boxtimes W_3$ and $W_1 \boxtimes (W_2 \boxtimes W_3)$ follows from showing that the ranges of homomorphisms (1.14) and (1.15) are equal to each other, which is of course obvious in the Lie algebra case since both (1.14) and (1.15) are isomorphisms to $(W_1 \otimes W_2 \otimes W_3)^*$. It turns out that, under this associativity isomorphism, (1.2) holds in both the Lie algebra case and the vertex algebra case; in the Lie algebra case, this is obvious because all the maps are the “tautological” ones.

Now we give the reader a preview of how, in the vertex algebra case, these compatibility and local grading restriction conditions on elements of $(W_1 \otimes W_2 \otimes W_3)^*$ will arise. As we have mentioned, in the Lie algebra case, an intertwining map from $W_1 \otimes W_2 \otimes W_3$ to W_4 corresponds to a module map from W_4^* to $(W_1 \otimes W_2 \otimes W_3)^*$. As was discussed in [H1], for the vertex operator algebra analogue, the image of any $w'_{(4)} \in W'_4$ under such an analogous map satisfies certain “compatibility” and “local grading restriction” conditions, and so these

conditions must be satisfied by those elements of $(W_1 \otimes W_2 \otimes W_3)^*$ lying in the ranges of the vertex-operator-algebra analogues of either of the maps (1.14) and (1.15) (or the maps (1.12) and (1.13)).

Besides these two conditions, satisfied by the elements of the ranges of the maps of both types (1.14) and (1.15), the elements of the ranges of the analogues of the homomorphisms (1.14) and (1.15) have their own separate properties. First note that any $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ induces the two maps

$$\begin{aligned} \mu_\lambda^{(1)} : W_1 &\rightarrow (W_2 \otimes W_3)^* \\ w_{(1)} &\mapsto \lambda(w_{(1)} \otimes \cdot \otimes \cdot) \end{aligned} \quad (1.16)$$

and

$$\begin{aligned} \mu_\lambda^{(2)} : W_3 &\rightarrow (W_1 \otimes W_2)^* \\ w_{(3)} &\mapsto \lambda(\cdot \otimes \cdot \otimes w_{(3)}). \end{aligned} \quad (1.17)$$

In the vertex operator algebra analogue [H1], if λ lies in the range of (1.14), then it must satisfy the condition that the elements $\mu_\lambda^{(1)}(w_{(1)})$ all lie in a suitable completion of the subspace $W_2 \boxtimes W_3$ of $(W_2 \otimes W_3)^*$, and if λ lies in the range of (1.15), then it must satisfy the condition that the elements $\mu_\lambda^{(2)}(w_{(3)})$ all lie in a suitable completion of the subspace $W_1 \boxtimes W_2$ of $(W_1 \otimes W_2)^*$. (Of course in the Lie algebra case, these statements are tautological.) In [H1], these important conditions, that $\mu_\lambda^{(1)}(W_1)$ lies in a suitable completion of $W_2 \boxtimes W_3$ and that $\mu_\lambda^{(2)}(W_3)$ lies in a suitable completion of $W_1 \boxtimes W_2$, are understood as “local grading restriction conditions” with respect to the two different ways of composing intertwining maps.

In the construction of our desired natural associativity isomorphism, since we want the ranges of (1.14) and (1.15) to be the same submodule of $(W_1 \otimes W_2 \otimes W_3)^*$, the ranges of both (1.14) and (1.15) should satisfy both of these conditions. This amounts to a certain “extension condition” in the vertex algebra case. When all these conditions are satisfied, it can in fact be proved [H1] that the associativity isomorphism does indeed exist and that in addition, the “associativity of intertwining maps” holds; that is, the “product” of two suitable intertwining maps can be written, in a certain sense, as the “iterate” of two suitable intertwining maps, and conversely. This equality of products with iterates, highly nontrivial in the vertex algebra case, amounts in the Lie algebra case to the easy statement that in the notation above, any intertwining map of the form $I_1 \circ (1_{W_1} \otimes I_2)$ can also be written as an intertwining map of the form $I^1 \circ (I^2 \otimes 1_{W_3})$, for a suitable “intermediate module” M_2 and suitable intertwining maps I^1 and I^2 , and conversely. The reason why this statement is easy in the Lie algebra case is that in fact *any* intertwining map F of the type (1.8) can be “factored” in either of these two ways; for example, to write F in the form $I_1 \circ (1_{W_1} \otimes I_2)$, take M_1 to be $W_2 \otimes W_3$, I_2 to be the canonical (identity) map and I_1 to be F itself (with the appropriate identifications having been made).

We are now ready to discuss the vertex algebra case.

1.2 The vertex algebra case

In this subsection, which should be carefully compared with the previous one, we shall lay out our “road map” of the constructions of the tensor product functors and the associativity isomorphisms for a suitable class of vertex algebras, generalizing, and also following the ideas of, the corresponding theory developed in [HL5], [HL6], [HL7] and [H1] for vertex operator algebras. Without yet specifying the precise class of vertex algebras that we shall be using in the body of this work, except to say that our vertex algebras will be \mathbb{Z} -graded and our modules will be \mathbb{C} -graded, we now discuss the vertex algebra case.

In this case, the concept of intertwining map involves the moduli space of Riemann spheres with one negatively oriented puncture and two positively oriented punctures and with local coordinates around each puncture; the details of the geometric structures needed in this theory are presented in [H4]. For each element of this moduli space there is a notion of intertwining map adapted to the particular element. Let z be a nonzero complex number and let $P(z)$ be the Riemann sphere $\hat{\mathbb{C}}$ with one negatively oriented puncture at ∞ and two positively oriented punctures at z and 0 , with local coordinates $1/w$, $w - z$ and w at these three punctures, respectively.

Let V be a vertex algebra (on which appropriate assumptions, including the existence of a suitable \mathbb{Z} -grading, will be made later), and let $Y(\cdot, x)$ be the vertex operator map defining the algebra structure (see Section 2 below for a brief summary of basic notions and notation, including the formal delta function). Let W_1 , W_2 and W_3 be modules for V , and let $Y_1(\cdot, x)$, $Y_2(\cdot, x)$ and $Y_3(\cdot, x)$ be the corresponding vertex operator maps. (The cases in which some of the W_i are V itself, and some of the Y_i are, correspondingly, Y , are important, but the most interesting cases are those where all three modules are different from V .) A “ $P(z)$ -intertwining map of type $\binom{W_3}{W_1 W_2}$ ” is a linear map

$$I : W_1 \otimes W_2 \longrightarrow \overline{W}_3, \quad (1.18)$$

where \overline{W}_3 is a certain completion of W_3 , related to its \mathbb{C} -grading, such that

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_3(v, x_1) I(w_{(1)} \otimes w_{(2)}) \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) I(Y_1(v, x_0) w_{(1)} \otimes w_{(2)}) \\ & \quad + x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) I(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \end{aligned} \quad (1.19)$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, where x_0 , x_1 and x_2 are commuting independent formal variables. This notion is motivated in detail in [HL4], [HL5] and [HL7]; we shall recall the motivation below.

Remark 1.3 In this theory, it is crucial to distinguish between formal variables and complex variables. Thus we shall use the following notational convention: *Throughout this work, unless we specify otherwise, the symbols x , x_0 , x_1 , x_2 , \dots , y , y_0 , y_1 , y_2 , \dots will denote*

commuting independent formal variables, and by contrast, the symbols z, z_0, z_1, z_2, \dots will denote complex numbers in specified domains, not formal variables.

Remark 1.4 Recall from [FHL] the definition of the notion of intertwining operator $\mathcal{Y}(\cdot, x)$ in the theory of vertex (operator) algebras. Given (W_1, Y_1) , (W_2, Y_2) and (W_3, Y_3) as above, an intertwining operator of type $\binom{W_3}{W_1 W_2}$ can be viewed as a certain type of linear map $\mathcal{Y}(\cdot, x)$ from $W_1 \otimes W_2$ to the vector space of formal series in x of the form $\sum_{n \in \mathbb{C}} w(n)x^n$, where the coefficients $w(n)$ lie in W_3 , and where we are allowing arbitrary complex powers of x , suitably “truncated from below” in this sum. The main property of an intertwining operator is the following “Jacobi identity”:

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_3(v, x_1) \mathcal{Y}(w_{(1)}, x_2) w_{(2)} \\ - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_{(1)}, x_2) Y_2(v, x_1) w_{(2)} \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y_1(v, x_0) w_{(1)}, x_2) w_{(2)} \end{aligned} \quad (1.20)$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. (When all three modules W_i are V itself and all four operators Y_i and \mathcal{Y} are Y itself, (1.20) becomes the usual Jacobi identity in the definition of the notion of vertex algebra. When W_1 is V , $W_2 = W_3$ and $\mathcal{Y} = Y_2 = Y_3$, (1.20) becomes the usual Jacobi identity in the definition of the notion of V -module.) The point is that by “substituting z for x_2 ” in (1.20), we obtain (1.19), where we make the identification

$$I(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, z) w_{(2)}; \quad (1.21)$$

the resulting complex powers of the complex number z are made precise by a choice of branch of the log function. The nonzero complex number z in the notion of $P(z)$ -intertwining map thus “comes from” the substitution of z for x_2 in the Jacobi identity in the definition of the notion of intertwining operator. In fact, this correspondence (given a choice of branch of log) actually defines an isomorphism between the space of $P(z)$ -intertwining maps and the space of intertwining operators of the same type ([HL5], [HL7]); this will be discussed below.

There is a natural linear injection

$$\text{Hom}(W_1 \otimes W_2, \overline{W}_3) \longrightarrow \text{Hom}(W'_3, (W_1 \otimes W_2)^*), \quad (1.22)$$

where here and below we denote by W' the (suitably defined) contragredient module of a V -module W ; we have $W'' = W$. Under this injection, a map $I \in \text{Hom}(W_1 \otimes W_2, \overline{W}_3)$ amounts to a map $I' : W'_3 \longrightarrow (W_1 \otimes W_2)^*$:

$$w'_{(3)} \mapsto \langle w'_{(3)}, I(\cdot \otimes \cdot) \rangle, \quad (1.23)$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between the contragredient of a module and its completion. If I is a $P(z)$ -intertwining map, then as in the Lie algebra case (see above),

where such a map is a module map, the map (1.23) intertwines two natural V -actions on W'_3 and $(W_1 \otimes W_2)^*$. We will see that in the present (vertex algebra) case, $(W_1 \otimes W_2)^*$ is typically not a V -module. The images of all the elements $w'_{(3)} \in W'_3$ under this map satisfy certain conditions, called the “ $P(z)$ -compatibility condition” and the “ $P(z)$ -local grading restriction condition,” as formulated in [HL5] and [HL7]; we shall discuss these below.

Given a category of V -modules and two modules W_1 and W_2 in this category, as in the Lie algebra case, the “ $P(z)$ -tensor product of W_1 and W_2 ” is then defined to be a pair (W_0, I_0) , where W_0 is a module in the category and I_0 is a $P(z)$ -intertwining map of type $\binom{W_0}{W_1 W_2}$, such that for any pair (W, I) with W a module in the category and I a $P(z)$ -intertwining map of type $\binom{W}{W_1 W_2}$, there is a unique morphism $\eta : W_0 \longrightarrow W$ such that $I = \bar{\eta} \circ I_0$; here and throughout this work we denote by $\bar{\chi}$ the linear map naturally extending a suitable linear map χ from a graded space to its appropriate completion. This universal property characterizes (W_0, I_0) up to canonical isomorphism, *if it exists*. We will denote the $P(z)$ -tensor product of W_1 and W_2 , if it exists, by $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$, and we will denote the image of $w_{(1)} \otimes w_{(2)}$ under $\boxtimes_{P(z)}$ by $w_{(1)} \boxtimes_{P(z)} w_{(2)}$, which is an element of $\overline{W_1 \boxtimes_{P(z)} W_2}$, not of $W_1 \boxtimes_{P(z)} W_2$.

From this definition and the natural map (1.22), we will see that if the $P(z)$ -tensor product of W_1 and W_2 exists, then its contragredient module can be realized as the union of ranges of all maps of the form (1.23) as W'_3 and I vary. Even if the $P(z)$ -tensor product of W_1 and W_2 does not exist, we denote this union (which is always a subspace stable under a natural action of V) by $W_1 \boxdot_{P(z)} W_2$. If the tensor product does exist, then

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxdot_{P(z)} W_2)', \quad (1.24)$$

$$W_1 \boxdot_{P(z)} W_2 = (W_1 \boxtimes_{P(z)} W_2)'; \quad (1.25)$$

examining (1.24) will show the reader why the notation \boxdot was chosen in the earlier papers ($\boxtimes = \boxdot'$). Several critical facts about $W_1 \boxdot_{P(z)} W_2$ were proved in [HL5], [HL6] and [HL7], notably, $W_1 \boxdot_{P(z)} W_2$ is equal to the subspace of $(W_1 \otimes W_2)^*$ consisting of all the elements satisfying the $P(z)$ -compatibility condition and the $P(z)$ -local grading restriction condition, and in particular, this subspace is V -stable; and the condition that $W_1 \boxdot_{P(z)} W_2$ is a module is equivalent to the existence of the $P(z)$ -tensor product $W_1 \boxtimes_{P(z)} W_2$. All these facts will be proved below.

In order to construct vertex tensor category structure, we need to construct appropriate natural associativity isomorphisms. Assuming the existence of the relevant tensor products, we in fact need to construct an appropriate natural isomorphism from $(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3$ to $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ for complex numbers z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. Note that we are using two distinct nonzero complex numbers, and that certain inequalities hold. This situation corresponds to the fact that a Riemann sphere with one negatively oriented puncture and three positively oriented punctures can be seen in two different ways as the “product” of two Riemann spheres each with one negatively oriented puncture and two positively oriented punctures; the detailed geometric motivation is presented in [H4], [HL4] and [H1].

To construct this natural isomorphism, we first consider compositions of certain inter-

twining maps. As we have mentioned, a $P(z)$ -intertwining map I of type $\binom{W_3}{W_1 W_2}$ maps into \overline{W}_3 rather than W_3 . Thus the existence of compositions of suitable intertwining maps always entails certain convergence. In particular, the existence of the composition $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ when $|z_1| > |z_2| > 0$ and the existence of the composition $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$ when $|z_2| > |z_1 - z_2| > 0$, for general elements $w_{(i)}$ of W_i , $i = 1, 2, 3$, requires the proof of certain convergence conditions. These conditions will be discussed in detail below.

Let us now assume these convergence conditions and let z_1, z_2 satisfy $|z_1| > |z_2| > |z_1 - z_2| > 0$. To construct the desired associativity isomorphism from $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ to $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$, it is equivalent (by duality) to give a suitable natural isomorphism from $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ to $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$. As we mentioned in the previous subsection, instead of constructing this isomorphism directly, we shall embed both of these spaces, separately, into the single space $(W_1 \otimes W_2 \otimes W_3)^*$.

We will see that $(W_1 \otimes W_2 \otimes W_3)^*$ carries a natural V -action analogous to the contragredient of the diagonal action in the Lie algebra case (recall the similar action of V on $(W_1 \otimes W_2)^*$ mentioned above). Also, for four V -modules W_1, W_2, W_3 and W_4 , we have a canonical notion of “ $P(z_1, z_2)$ -intertwining map from $W_1 \otimes W_2 \otimes W_3$ to \overline{W}_4 ” given by a vertex-algebraic analogue of (1.9); for this notion, we need only that z_1 and z_2 are nonzero and distinct. The relation between these two concepts comes from the natural linear injection

$$\begin{aligned} \text{Hom}(W_1 \otimes W_2 \otimes W_3, \overline{W}_4) &\longrightarrow \text{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*) \\ F &\mapsto F', \end{aligned} \quad (1.26)$$

where $F' : W'_4 \longrightarrow (W_1 \otimes W_2 \otimes W_3)^*$ is given by

$$\nu \mapsto \nu \circ F, \quad (1.27)$$

which is indeed well defined. Under this natural map, the $P(x_1, z_2)$ -intertwining maps correspond precisely to the maps from W'_4 to $(W_1 \otimes W_2 \otimes W_3)^*$ that intertwine the two natural V -actions on W'_4 and $(W_1 \otimes W_2 \otimes W_3)^*$.

Now for modules W_1, W_2, W_3, W_4, M_1 , and a $P(z_1)$ -intertwining map I_1 and a $P(z_2)$ -intertwining map I_2 of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, where $|z_1| > |z_2| > 0$, it turns out that by definition the composition $\bar{I}_1 \circ (1_{W_1} \otimes I_2)$ is a $P(z_1, z_2)$ -intertwining map. Similarly, for a $P(z_2)$ -intertwining map I^1 and a $P(z_1 - z_2)$ -intertwining map I^2 of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, where $|z_2| > |z_1 - z_2| > 0$, the composition $\bar{I}^1 \circ (I^2 \otimes 1_{W_3})$ is a $P(z_1, z_2)$ -intertwining map. Hence we have two maps intertwining the V -actions:

$$\begin{aligned} W'_4 &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto \nu \circ F_1, \end{aligned} \quad (1.28)$$

where F_1 is the intertwining map $\bar{I}_1 \circ (1_{W_1} \otimes I_2)$, and

$$\begin{aligned} W'_4 &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto \nu \circ F_2, \end{aligned} \quad (1.29)$$

where F_2 is the intertwining map $\bar{I}^1 \circ (I^2 \circ 1_{W_3})$.

It is important to note that we can express these compositions $\bar{I}_1 \circ (1_{W_1} \otimes I_2)$ and $\bar{I}^1 \circ (I^2 \otimes 1_{W_3})$ in terms of intertwining operators, as discussed in Remark 1.4. Let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}^1$ and \mathcal{Y}^2 be the intertwining operators corresponding to I_1, I_2, I^1 and I^2 , respectively. Then the compositions $\bar{I}_1 \circ (1_{W_1} \otimes I_2)$ and $\bar{I}^1 \circ (I^2 \otimes 1_{W_3})$ correspond to the “product” $\mathcal{Y}_1(\cdot, x_1)\mathcal{Y}_2(\cdot, x_2)\cdot$ and “iterate” $\mathcal{Y}^1(\mathcal{Y}^2(\cdot, x_0)\cdot, x_2)\cdot$ of intertwining operators, respectively, and we make the “substitutions” (in the sense of Remark 1.4) $x_1 \mapsto z_1, x_2 \mapsto z_2$ and $x_0 \mapsto z_1 - z_2$ in order to express the two compositions of intertwining maps as the “product” $\mathcal{Y}_1(\cdot, z_1)\mathcal{Y}_2(\cdot, z_2)\cdot$ and “iterate” $\mathcal{Y}^1(\mathcal{Y}^2(\cdot, z_1 - z_2)\cdot, z_2)\cdot$ of intertwining maps, respectively. (These products and iterates involve a branch of the log function and also certain convergence.)

Just as in the Lie algebra case, the special cases in which the modules W_4 are two iterated tensor product modules and the “intermediate” modules M_1 and M_2 are two tensor product modules are particularly interesting: When $W_4 = W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ and $M_1 = W_2 \boxtimes_{P(z_2)} W_3$, and I_1 and I_2 are the corresponding canonical intertwining maps, (1.28) gives the natural V -homomorphism

$$\begin{aligned} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \\ &\nu(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}))); \end{aligned} \quad (1.30)$$

when $W_4 = (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ and $M_2 = W_1 \boxtimes_{P(z_1-z_2)} W_2$, and I^1 and I^2 are the corresponding canonical intertwining maps, (1.29) gives the natural V -homomorphism

$$\begin{aligned} (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\ \nu &\mapsto (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \\ &\nu((w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)})). \end{aligned} \quad (1.31)$$

It turns out that both of these maps are injections [H1], as we prove below, so that we are embedding the spaces $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ and $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ into the space $(W_1 \otimes W_2 \otimes W_3)^*$. Following the ideas in [H1], we shall give a precise description of the ranges of these two maps, and under suitable conditions, prove that the two ranges are the same; this will establish the associativity isomorphism.

More precisely, as in [H1], we prove that for any $P(z_1, z_2)$ -intertwining map F , the image of any $\nu \in W'_4$ under F' (recall (1.27)) satisfies certain conditions that we call the “ $P(z_1, z_2)$ -compatibility condition” and the “ $P(z_1, z_2)$ -local grading restriction condition.” Hence, as special cases, the elements of $(W_1 \otimes W_2 \otimes W_3)^*$ in the ranges of either of the maps (1.28) or (1.29), and in particular, of (1.30) or (1.31), satisfy these conditions.

In addition, any $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ induces two maps $\mu_\lambda^{(1)}$ and $\mu_\lambda^{(2)}$ as in (1.16) and (1.17). We will see that any element λ of the range of (1.28), and in particular, of (1.30), must satisfy the condition that the elements $\mu_\lambda^{(1)}(w_{(1)})$ all lie in a suitable completion of

the subspace $W_2 \boxtimes_{P(z_2)} W_3$ of $(W_2 \otimes W_3)^*$, and any element λ of the range of (1.29), and in particular, of (1.31), must satisfy the condition that the elements $\mu_\lambda^{(2)}(w_{(3)})$ all lie in a suitable completion of the subspace $W_1 \boxtimes_{P(z_1-z_2)} W_2$ of $(W_1 \otimes W_2)^*$. These conditions will be called the “ $P^{(1)}(z)$ -local grading restriction condition” and the “ $P^{(2)}(z)$ -local grading restriction condition,” respectively.

It turns out that the construction of the desired natural associativity isomorphism follows from showing that the ranges of both of (1.30) and (1.31) satisfy both of these conditions. This amounts to a certain “extension condition” on our module category. When this extension condition and a suitable convergence condition are satisfied, as in [H1] we show below that the desired associativity isomorphisms do exist, and that in addition, the associativity of intertwining maps holds. That is, let z_1 and z_2 be complex numbers satisfying the inequalities $|z_1| > |z_2| > |z_1 - z_2| > 0$. Then for any $P(z_1)$ -intertwining map I_1 and $P(z_2)$ -intertwining map I_2 of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, there is a suitable module M_2 , and a $P(z_2)$ -intertwining map I^1 and a $P(z_1 - z_2)$ -intertwining map I^2 of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, such that

$$\langle w'_{(4)}, \bar{I}_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)})) \rangle = \langle w'_{(4)}, \bar{I}^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}) \rangle \quad (1.32)$$

for $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$; and conversely, given I^1 and I^2 as indicated, there exist a suitable module M_1 and maps I_1 and I_2 with the indicated properties. In terms of intertwining operators (recall the comments above), the equality (1.32) reads

$$\begin{aligned} & \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle|_{x_1=z_1, x_2=z_2} \\ &= \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle|_{x_0=z_1-z_2, x_2=z_2}, \end{aligned} \quad (1.33)$$

where $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}^1$ and \mathcal{Y}^2 are the intertwining operators corresponding to I_1, I_2, I^1 and I^2 , respectively. (As we have been mentioning, the substitution of complex numbers for formal variables involves a branch of the log function and also certain convergence.) In this sense, the associativity asserts that the “product” of two suitable intertwining maps can be written as the “iterate” of two suitable intertwining maps, and conversely.

From this construction of the natural associativity isomorphisms we will see, by analogy with (1.2), that $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$ is mapped naturally to $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$. The coherence property of the associativity isomorphisms will follow from this fact.

Remark 1.5 Note that equation (1.33) can be written as

$$\mathcal{Y}_1(w_{(1)}, z_1) \mathcal{Y}_2(w_{(2)}, z_2) = \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_1 - z_2) w_{(2)}, z_2), \quad (1.34)$$

with the appearance of the complex numbers being understood as substitutions in the sense mentioned above, and with the “generic” vectors $w_{(3)}$ and $w'_{(4)}$ being implicit. This (rigorous) equation amounts to the “operator product expansion” in the physics literature on conformal field theory; indeed, in our language, if we expand the right-hand side of (1.34) in powers of $z_1 - z_2$, we find that a product of intertwining maps is expressed as an expansion in

powers of $z_1 - z_2$, with coefficients that are again intertwining maps, of the form $\mathcal{Y}^1(w, z_2)$. When all three modules are the vertex algebra itself, and all the intertwining operators are the canonical vertex operator $Y(\cdot, x)$ itself, this “operator product expansion” follows easily from the Jacobi identity. But for intertwining operators in general, it is hard to prove the operator product expansion, that is, to prove the assertions involving (1.32) and (1.33) above.

Remark 1.6 The constructions of the tensor product modules and of the associativity isomorphisms previewed above for suitably general vertex algebras follow those in [HL5], [HL6], [HL7] and [H1]. Alternative constructions are certainly possible. For example, an alternative construction of the tensor product modules was given in [Li]. However, no matter what construction is used for the tensor product modules of suitably general vertex algebras, one cannot avoid constructing structures and proving results equivalent to what is carried out in this work. The constructions in this work of the tensor product functors and of the natural associativity isomorphisms are crucial in the deeper part of the theory of vertex tensor categories.

Remark 1.7 A braided tensor category structure on certain module categories for affine Lie algebras, and more generally, on certain module categories for “chiral algebras” associated with “rational conformal field theories,” was discovered by Moore and Seiberg [MS] in their important study of conformal field theory. However, they constructed this structure based on the assumption of the existence of a suitable tensor product functor (including a tensor product module) and also the assumption of the existence of a suitable operator product expansion for chiral vertex operators, which is essentially equivalent to assuming the associativity of intertwining maps, as we have expressed it above. As we have discussed, the desired tensor product modules were constructed under suitable conditions in the series of papers [HL5], [HL6] and [HL7], and in [H1] the appropriate natural associativity isomorphisms among tensor products of triples of modules were constructed and it was shown that this is equivalent to the desired associativity of intertwining maps (and thus the existence of a suitable operator product expansion). The results in these papers will now be generalized in this work.

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2 The setting

In this section we define and discuss the basic structures and introduce some notation that will be used in this work. More specifically, we first introduce the notions of “conformal vertex algebra” and “Möbius vertex algebra.” A conformal vertex algebra is just a vertex algebra equipped with a conformal vector satisfying the usual axioms; a Möbius vertex algebra is a variant of a “quasi-vertex operator algebra” as in [FHL], with the difference that the two grading restriction conditions in the definition of vertex operator algebra are not required. We then define the notion of module for each of these types of vertex algebra. Relaxing the $L(0)$ -semisimplicity in the definition of module we obtain the notion of “generalized module.” Finally, we notice that in order to have a contragredient functor on the module category under consideration, we need to impose a stronger grading condition. This leads to the notions of “strong gradedness” of Möbius vertex algebras and their generalized modules. In this work we are mainly interested in certain full subcategories of the category of strongly graded generalized modules for certain strongly graded Möbius vertex algebras. Throughout the work we shall assume some familiarity with the material in [FLM2], [FHL] and [LL], and in fact, as we mentioned in the Introduction, we will not require any results in vertex (operator) algebra theory outside of these three books.

Throughout, we shall use the notation \mathbb{N} for the nonnegative integers and \mathbb{Z}_+ for the positive integers.

We shall continue to use the notational convention concerning formal variables and complex variables given in Remark 1.3. Recall from [FLM2], [FHL] or [LL] that the “formal delta function” is defined as the formal Laurent series

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$

We will consistently use the *binomial expansion convention*: For any complex number λ , $(x + y)^\lambda$ is to be expanded in nonnegative integral powers of the second variable, i.e.,

$$(x + y)^\lambda = \sum_{n \in \mathbb{N}} \binom{\lambda}{n} x^{\lambda-n} y^n.$$

Here x or y might be something other than a formal variable (or a nonzero complex multiple of a formal variable); for instance, x or y (but not both!) might be a nonzero complex number, or x or y might be some more complicated object. The use of the binomial expansion convention will be clear in context.

Objects like $\delta(x)$ and $(x + y)^\lambda$ lie in spaces of formal series. Some of the spaces that we will use are, with W a vector space (over \mathbb{C}) and x a formal variable:

$$W[x] = \left\{ \sum_{n \in \mathbb{N}} a_n x^n \mid a_n \in W, \text{ all but finitely many } a_n = 0 \right\},$$

$$W[x, x^{-1}] = \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \mid a_n \in W, \text{ all but finitely many } a_n = 0 \right\},$$

$$\begin{aligned}
W[[x]] &= \left\{ \sum_{n \in \mathbb{N}} a_n x^n \mid a_n \in W \text{ (with possibly infinitely many } a_n \text{ not 0)} \right\}, \\
W((x)) &= \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \mid a_n \in W, a_n = 0 \text{ for sufficiently small } n \right\}, \\
W[[x, x^{-1}]] &= \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \mid a_n \in W \text{ (with possibly infinitely many } a_n \text{ not 0)} \right\}.
\end{aligned}$$

We will also need

$$W\{x\} = \left\{ \sum_{n \in \mathbb{C}} a_n x^n \mid a_n \in W \text{ for } n \in \mathbb{C} \right\} \quad (2.1)$$

as in [FLM2]; here the powers of the formal variable are complex, and the coefficients may all be nonzero. We will also use analogues of these spaces involving two or more formal variables.

The following formal version of Taylor's theorem is easily verified by direct expansion (see Proposition 8.3.1 of [FLM2]): For $f(x) \in W\{x\}$,

$$e^{y \frac{d}{dx}} f(x) = f(x + y), \quad (2.2)$$

where the exponential denotes the formal exponential series, and where we are using the binomial expansion convention on the right-hand side. It is important to note that this formula holds for arbitrary formal series $f(x)$ with complex powers of x , where $f(x)$ need not be an expansion in any sense of an analytic function (again, see Proposition 8.3.1 of [FLM2]).

The formal delta function $\delta(x)$ has the following simple and fundamental property: For any $f(x) \in W[x, x^{-1}]$,

$$f(x)\delta(x) = f(1)\delta(x). \quad (2.3)$$

(Here we are taking the liberty of writing complex numbers to the right of vectors in W .) This is proved immediately by observing its truth for $f(x) = x^n$ and then using linearity. This property has many important variants; in general, whenever an expression is multiplied by the formal delta function, we may formally set the argument appearing in the delta function equal to 1, provided that the relevant algebraic expressions make sense. For example, for any

$$X(x_1, x_2) \in (\text{End } W)[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$$

such that

$$\lim_{x_1 \rightarrow x_2} X(x_1, x_2) = X(x_1, x_2) \Big|_{x_1=x_2} \quad (2.4)$$

exists, we have

$$X(x_1, x_2)\delta\left(\frac{x_1}{x_2}\right) = X(x_2, x_2)\delta\left(\frac{x_1}{x_2}\right). \quad (2.5)$$

The existence of the “algebraic limit” defined in (2.4) means that for an arbitrary vector $w \in W$, the coefficient of each power of x_2 in the formal expansion $X(x_1, x_2)w \Big|_{x_1=x_2}$ is a finite

sum. In general, the existence of such “algebraic limits,” and also such products of formal sums, always means that the coefficient of each monomial in the relevant formal variables gives a finite sum. Often, proving the existence of the relevant algebraic limits (or products) is a much more subtle matter than computing such limits (or products), just as in analysis. (In this work, we will typically use “substitution notation” like $\Big|_{x_1=x_2}$ or $X(x_2, x_2)$ rather than the formal limit notation on the left-hand side of (2.4).) Below, we will give a more sophisticated analogue of the delta-function substitution principle (2.5), an analogue that we will need in this work.

This analogue, and in fact, many fundamental principles of vertex operator algebra theory, are based on certain delta-function expressions of the following type, involving three (commuting and independent, as usual) formal variables:

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) = \sum_{n \in \mathbb{Z}} \frac{(x_1 - x_2)^n}{x_0^{n+1}} = \sum_{m \in \mathbb{N}, n \in \mathbb{Z}} (-1)^m \binom{n}{m} x_0^{-n-1} x_1^{n-m} x_2^m;$$

here the binomial expansion convention is of course being used.

The following important identities involving such three-variable delta-function expressions are easily proved (see [FLM2] or [LL], where extensive motivation for these formulas is also given):

$$x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) = x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right), \quad (2.6)$$

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right). \quad (2.7)$$

Note that the three terms in (2.7) involve nonnegative integral powers of x_2 , x_1 and x_0 , respectively. In particular, the two terms on the left-hand side of (2.7) are unequal formal Laurent series in three variables, even though they might appear equal at first glance. We shall use these two identities extensively.

Remark 2.1 Here is the useful analogue, mentioned above, of the delta-function substitution principle (2.5): Let

$$f(x_1, x_2, y) \in (\text{End } W)[[x_1, x_1^{-1}, x_2, x_2^{-1}, y, y^{-1}]] \quad (2.8)$$

be such that

$$\lim_{x_1 \rightarrow x_2} f(x_1, x_2, y) \text{ exists} \quad (2.9)$$

and such that for any $w \in W$,

$$f(x_1, x_2, y)w \in W[[x_1, x_1^{-1}, x_2, x_2^{-1}]]((y)). \quad (2.10)$$

Then

$$x_1^{-1} \delta \left(\frac{x_2 - y}{x_1} \right) f(x_1, x_2, y) = x_1^{-1} \delta \left(\frac{x_2 - y}{x_1} \right) f(x_2 - y, x_2, y). \quad (2.11)$$

For this principle, see Remark 2.3.25 of [LL], where the proof is also presented.

The following formal residue notation will be useful: For

$$f(x) = \sum_{n \in \mathbb{C}} a_n x^n \in W\{x\}$$

(note that the powers of x need not be integral),

$$\text{Res}_x f(x) = a_{-1}.$$

For instance, for the expression in (2.6),

$$\text{Res}_{x_2} x_2^{-1} \left(\frac{x_1 - x_0}{x_2} \right) = 1. \quad (2.12)$$

For a vector space W , we will denote its vector space dual by W^* ($= \text{Hom}_{\mathbb{C}}(W, \mathbb{C})$), and we will use the notation $\langle \cdot, \cdot \rangle_W$, or $\langle \cdot, \cdot \rangle$ if the underlying space W is clear, for the canonical pairing between W^* and W .

We will use the following version of the notion of “conformal vertex algebra”: A conformal vertex algebra is a vertex algebra (in the sense of Borchers [B]; see [LL]) equipped with a \mathbb{Z} -grading and with a conformal vector satisfying the usual compatibility conditions. Specifically:

Definition 2.2 A *conformal vertex algebra* is a \mathbb{Z} -graded vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)} \quad (2.13)$$

(for $v \in V_{(n)}$, we say the *weight* of v is n and we write $\text{wt } v = n$) equipped with a linear map $V \otimes V \rightarrow V[[x, x^{-1}]]$, or equivalently,

$$\begin{aligned} V &\rightarrow (\text{End } V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \text{End } V), \end{aligned} \quad (2.14)$$

$Y(v, x)$ denoting the *vertex operator associated with* v , and equipped also with two distinguished vectors $\mathbf{1} \in V_{(0)}$ (the *vacuum vector*) and $\omega \in V_{(2)}$ (the *conformal vector*), satisfying the following conditions for $u, v \in V$: the *lower truncation condition*:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large} \quad (2.15)$$

(or equivalently, $Y(u, x)v \in V((x))$); the *vacuum property*:

$$Y(\mathbf{1}, x) = 1_V; \quad (2.16)$$

the *creation property*:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v \quad (2.17)$$

(that is, $Y(v, x)\mathbf{1}$ involves only nonnegative integral powers of x and the constant term is v); the *Jacobi identity* (the main axiom):

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2) \end{aligned} \quad (2.18)$$

(note that when each expression in (2.18) is applied to any element of V , the coefficient of each monomial in the formal variables is a finite sum; on the right-hand side, the notation $Y(\cdot, x_2)$ is understood to be extended in the obvious way to $V[[x_0, x_0^{-1}]]$); the *Virasoro algebra relations*:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m, 0}c \quad (2.19)$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e., } Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}, \quad (2.20)$$

$$c \in \mathbb{C} \quad (2.21)$$

(the *central charge* or *rank* of V);

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x) \quad (2.22)$$

(the $L(-1)$ -*derivative property*); and

$$L(0)v = nv = (\text{wt } v)v \quad \text{for } n \in \mathbb{Z} \quad \text{and } v \in V_{(n)}. \quad (2.23)$$

This completes the definition of the notion of conformal vertex algebra. We will denote such a conformal vertex algebra by $(V, Y, \mathbf{1}, \omega)$ or simply by V .

The only difference between the definition of conformal vertex algebra and the definition of *vertex operator algebra* (in the sense of [FLM2] and [FHL]) is that a vertex operator algebra V also satisfies the two *grading restriction conditions*

$$V_{(n)} = 0 \quad \text{for } n \text{ sufficiently negative,} \quad (2.24)$$

and

$$\dim V_{(n)} < \infty \quad \text{for } n \in \mathbb{Z}. \quad (2.25)$$

(As we mentioned above, a *vertex algebra* is the same thing as a conformal vertex algebra but without the assumptions of a grading or a conformal vector, or, of course, the $L(n)$'s.)

Remark 2.3 Of course, not every vertex algebra is conformal. For example, it is well known [B] that any commutative associative algebra A with unit 1, together with a derivation $D : A \rightarrow A$ can be equipped with a vertex algebra structure, by:

$$Y(\cdot, x)\cdot : A \times A \rightarrow A[[x]], \quad Y(a, x)b = (e^{xD}a)b,$$

and $\mathbf{1} = 1$. In particular, $u_n = 0$ for any $u \in A$ and $n \geq 0$. If ω is a conformal vector for such a vertex algebra, then for any $u \in A$, $Du = u_{-2}\mathbf{1} = L(-1)u$ from (2.17) and (2.22), so $D = L(-1) = \omega_0$, which equals 0 because $\omega = L(0)\omega/2 = \omega_1\omega/2 = 0$. Thus a vertex algebra constructed from a commutative associative algebra with nonzero derivation in this way cannot be conformal.

Remark 2.4 The theory of vertex tensor categories inherently uses the whole moduli space of spheres with two positively oriented punctures and one negatively oriented puncture (and in fact, more generally, with arbitrary numbers of positively oriented punctures and one negatively oriented puncture) equipped with general (analytic) local coordinates vanishing at the punctures. Because of the analytic local coordinates, our constructions require certain conditions on the Virasoro operators. However, recalling the definition of the moduli space elements $P(z)$ from Section 1.2, we point out that if we restrict our attention to elements of the moduli space of only the type $P(z)$, then the relevant operations of sewing and subsequently decomposing Riemann spheres continue to yield spheres of the same type, and rather than general conformal transformations around the punctures, only Möbius (projective) transformations around the punctures are needed. This makes it possible to develop the essential structure of our tensor product theory by working entirely with spheres of this special type; the general vertex tensor category theory then follows from the structure thus developed. This is why, in the present work, we are focusing on the theory of $P(z)$ -tensor products. Correspondingly, it turns out that it is very natural for us to consider, along with the notion of conformal vertex algebra (Definition 2.2), a weaker notion of vertex algebra involving only the three-dimensional subalgebra of the Virasoro algebra corresponding to the group of Möbius transformations. That is, instead of requiring an action of the whole Virasoro algebra, we use only the action of the Lie algebra $\mathfrak{sl}(2)$ generated by $L(-1)$, $L(0)$ and $L(1)$. Thus we get a notion essentially identical to the notion of “quasi-vertex operator algebra” in [FHL]; the reason for focusing on this notion here is the same as the reason why it was considered in [FHL]. Here we designate this notion by the term “Möbius vertex algebra”; the only difference between the definition of Möbius vertex algebra and the definition of quasi-vertex operator algebra [FHL] is that a quasi-vertex operator algebra V also satisfies the two grading restriction conditions (2.24) and (2.25).

Thus we formulate:

Definition 2.5 The notion of *Möbius vertex algebra* is defined in the same way as that of conformal vertex algebra except that in addition to the data and axioms concerning V , Y and $\mathbf{1}$ (through (2.18) in Definition 2.2), we assume (in place of the existence of the conformal vector ω and the Virasoro algebra conditions (2.19), (2.20) and (2.21)) the following: We have a representation ρ of $\mathfrak{sl}(2)$ on V given by

$$L(j) = \rho(L_j), \quad j = 0, \pm 1, \quad (2.26)$$

where $\{L_{-1}, L_0, L_1\}$ is a basis of $\mathfrak{sl}(2)$ with Lie brackets

$$[L_0, L_{-1}] = L_{-1}, \quad [L_0, L_1] = -L_1, \quad \text{and} \quad [L_{-1}, L_1] = -2L_0, \quad (2.27)$$

and the following conditions hold for $v \in V$:

$$[L(-1), Y(v, x)] = Y(L(-1)v, x), \quad (2.28)$$

$$[L(0), Y(v, x)] = Y(L(0)v, x) + xY(L(-1)v, x), \quad (2.29)$$

$$[L(1), Y(v, x)] = Y(L(1)v, x) + 2xY(L(0)v, x) + x^2Y(L(-1)v, x), \quad (2.30)$$

and also, (2.22) and (2.23). Of course, (2.28)–(2.30) can be written as

$$\begin{aligned} [L(j), Y(v, x)] &= \sum_{k=0}^{j+1} \binom{j+1}{k} x^k Y(L(j-k)v, x) \\ &= \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} Y(L(k-1)v, x) \end{aligned} \quad (2.31)$$

for $j = 0, \pm 1$.

We will denote such a Möbius vertex algebra by $(V, Y, \mathbf{1}, \rho)$ or simply by V . Note that there is no notion of central charge (or rank) for a Möbius vertex algebra. Also, a conformal vertex algebra can certainly be viewed as a Möbius vertex algebra in the obvious way. (Of course, a conformal vertex algebra could have other $\mathfrak{sl}(2)$ -structures making it a Möbius vertex algebra in a different way.)

Remark 2.6 By (2.26) and (2.27) we have $[L(0), L(j)] = -jL(j)$ for $j = 0, \pm 1$. Hence

$$L(j)V_{(n)} \subset V_{(n-j)}, \quad \text{for } j = 0, \pm 1. \quad (2.32)$$

Moreover, from (2.28), (2.29) and (2.30) with $v = \mathbf{1}$ we get, by (2.16) and (2.17),

$$L(j)\mathbf{1} = 0 \quad \text{for } j = 0, \pm 1.$$

Remark 2.7 Not every Möbius vertex algebra is conformal. As an example, take the commutative associative algebra $\mathbb{C}[t]$ with derivation $D = -d/dt$, and form a vertex algebra as in Remark 2.3. By Remark 2.3, this vertex algebra is not conformal. However, define linear operators

$$L(-1) = D, \quad L(0) = tD, \quad L(1) = t^2D$$

on $\mathbb{C}[t]$. Then it is straightforward to verify that $\mathbb{C}[t]$ becomes a Möbius vertex algebra with these operators giving a representation of $\mathfrak{sl}(2)$ having the desired properties and with the \mathbb{Z} -grading (by nonpositive integers) given by the eigenspace decomposition with respect to $L(0)$.

Remark 2.8 It is also easy to see that not every vertex algebra is Möbius. For example, take the two-dimensional commutative associative algebra $A = \mathbb{C}1 \oplus \mathbb{C}a$ with 1 as identity and $a^2 = 0$. The linear operator D defined by $D(1) = 0$, $D(a) = a$ is a nonzero derivation of A . Hence A has a vertex algebra structure by Remark 2.3. Now if it is a module for $\mathfrak{sl}(2)$ as in Definition 2.5, since A is two-dimensional and $L(0)1 = 0$, $L(0)$ must act as 0. But then $D = L(-1) = [L(0), L(-1)] = 0$, a contradiction.

A module for a conformal vertex algebra V is a module for V viewed as a vertex algebra such that the conformal element acts in the same way as in the definition of vertex operator algebra. More precisely:

Definition 2.9 Given a conformal vertex algebra $(V, Y, \mathbf{1}, \omega)$, a *module* for V is a \mathbb{C} -graded vector space

$$W = \coprod_{n \in \mathbb{C}} W_{(n)} \quad (2.33)$$

(graded by *weights*) equipped with a linear map $V \otimes W \rightarrow W[[x, x^{-1}]]$, or equivalently,

$$\begin{aligned} V &\rightarrow (\text{End } W)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \text{End } W) \end{aligned} \quad (2.34)$$

(note that the sum is over \mathbb{Z} , not \mathbb{C}), $Y(v, x)$ denoting the *vertex operator on W associated with v* , such that all the defining properties of a conformal vertex algebra that make sense hold. That is, the following conditions are satisfied: the lower truncation condition: for $v \in V$ and $w \in W$,

$$v_n w = 0 \quad \text{for } n \text{ sufficiently large} \quad (2.35)$$

(or equivalently, $Y(v, x)w \in W((x))$); the vacuum property:

$$Y(\mathbf{1}, x) = 1_W; \quad (2.36)$$

the Jacobi identity for vertex operators on W : for $u, v \in V$,

$$\begin{aligned} x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2) Y(u, x_1) \\ = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2) \end{aligned} \quad (2.37)$$

(note that on the right-hand side, $Y(u, x_0)$ is the operator on V associated with u); the Virasoro algebra relations on W with scalar c equal to the central charge of V :

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m, 0}c \quad (2.38)$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e., } Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}; \quad (2.39)$$

$$\frac{d}{dx} Y(v, x) = Y(L(-1)v, x) \quad (2.40)$$

(the $L(-1)$ -derivative property); and

$$(L(0) - n)w = 0 \quad \text{for } n \in \mathbb{C} \text{ and } w \in W_{(n)}. \quad (2.41)$$

This completes the definition of the notion of module for a conformal vertex algebra.

Remark 2.10 The Virasoro algebra relations (2.38) for a module action follow from the corresponding relations (2.19) for V together with the Jacobi identities (2.18) and (2.37) and the $L(-1)$ -derivative properties (2.22) and (2.40), as we recall from (for example) [FHL] or [LL].

We also have:

Definition 2.11 The notion of *module* for a Möbius vertex algebra is defined in the same way as that of module for a conformal vertex algebra except that in addition to the data and axioms concerning W and Y (through (2.37) in Definition 2.9), we assume (in place of the Virasoro algebra conditions (2.38) and (2.39)) a representation ρ of $\mathfrak{sl}(2)$ on W given by (2.26) and the conditions (2.28), (2.29) and (2.30), for operators acting on W , and also, (2.40) and (2.41).

In addition to modules, we have the following notion of *generalized module* (or *logarithmic module*, as in, for example, [Mi]):

Definition 2.12 A *generalized module* for a conformal (respectively, Möbius) vertex algebra is defined in the same way as a module for a conformal (respectively, Möbius) vertex algebra except that in the grading (2.33), each space $W_{(n)}$ is replaced by $W_{[n]}$, where $W_{[n]}$ is the generalized $L(0)$ -eigenspace corresponding to the (generalized) eigenvalue $n \in \mathbb{C}$; that is, (2.33) and (2.41) in the definition are replaced by

$$W = \coprod_{n \in \mathbb{C}} W_{[n]} \quad (2.42)$$

and

$$\text{for } n \in \mathbb{C} \text{ and } w \in W_{[n]}, (L(0) - n)^m w = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large,} \quad (2.43)$$

respectively. For $w \in W_{[n]}$, we still write $\text{wt } w = n$ for the (generalized) weight of w .

We will denote such a module or generalized module just defined by (W, Y) , or sometimes by (W, Y_W) or simply by W . We will use the notation

$$\pi_n : W \rightarrow W_{[n]} \quad (2.44)$$

for the projection from W to its subspace of (generalized) weight n , and for its natural extensions to spaces of formal series with coefficients in W . In either the conformal or Möbius case, a module is of course a generalized module.

Remark 2.13 For any vector space U on which an operator, say, $L(0)$, acts in such a way that

$$U = \coprod_{n \in \mathbb{C}} U_{[n]} \quad (2.45)$$

where for $n \in \mathbb{C}$,

$$U_{[n]} = \{u \in U \mid (L(0) - n)^m u = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large}\},$$

we shall typically use the same projection notation

$$\pi_n : U \rightarrow U_{[n]} \quad (2.46)$$

as in (2.44). If instead of (2.45) we have only

$$U = \sum_{n \in \mathbb{C}} U_{[n]},$$

then in fact this sum is indeed direct, and for any $L(0)$ -stable subspace T of U , we have

$$T = \coprod_{n \in \mathbb{C}} T_{[n]}$$

(as with ordinary rather than generalized eigenspaces).

Remark 2.14 A module for a conformal vertex algebra V is obviously again a module for V viewed as a Möbius vertex algebra, and conversely, a module for V viewed as a Möbius vertex algebra is a module for V viewed as a conformal vertex algebra, by Remark 2.10. Similarly, the generalized modules for a conformal vertex algebra V are exactly the generalized modules for V viewed as a Möbius vertex algebra.

Remark 2.15 A conformal or Möbius vertex algebra is a module for itself (and in particular, a generalized module for itself).

Remark 2.16 In either the conformal or Möbius vertex algebra case, we have the obvious notions of *V-module homomorphism*, *submodule*, *quotient module*, and so on; in particular, homomorphisms are understood to be grading-preserving. We sometimes write the vector space of (generalized-) module maps (homomorphisms) $W_1 \rightarrow W_2$ for (generalized) V -modules W_1 and W_2 as $\text{Hom}_V(W_1, W_2)$.

Remark 2.17 We have chosen the name “generalized module” here because the vector space underlying the module is graded by generalized eigenvalues. (This notion is different from the notion of “generalized module” used in [HL5]. A generalized module for a vertex operator algebra V as defined in, for example, Definition 2.11 of [HL5] is precisely a module for V viewed as a conformal vertex algebra.)

We will use the following notion of (formal algebraic) completion of a generalized module:

Definition 2.18 Let $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ be a generalized module for a Möbius (or conformal) vertex algebra. We denote by \overline{W} the (formal) completion of W with respect to the \mathbb{C} -grading, that is,

$$\overline{W} = \prod_{n \in \mathbb{C}} W_{[n]}. \quad (2.47)$$

We will use the same notation \overline{U} for any \mathbb{C} -graded subspace U of W . We will continue to use the notation π_n for the projection from \overline{W} to $W_{[n]}$. We will also continue to use the notation $\langle \cdot, \cdot \rangle_W$, or $\langle \cdot, \cdot \rangle$ if the underlying space is clear, for the canonical pairing between the subspace $\coprod_{n \in \mathbb{C}} (W_{[n]})^*$ of W^* , and \overline{W} . We are of course viewing $(W_{[n]})^*$ as embedded in W^* in the natural way, that is, for $w^* \in (W_{[n]})^*$,

$$\langle w^*, w \rangle_W = \langle w^*, w_n \rangle_{W_{[n]}} \quad (2.48)$$

for any $w = \sum_{m \in \mathbb{C}} w_m$ (finite sum) in W , where $w_m \in W_{[m]}$.

The following weight formula holds for generalized modules, generalizing the corresponding formula in the module case (cf. [Mi]):

Proposition 2.19 *Let W be a generalized module for a Möbius (or conformal) vertex algebra V . Let both $v \in V$ and $w \in W$ be homogeneous. Then*

$$\text{wt}(v_n w) = \text{wt } v + \text{wt } w - n - 1 \quad \text{for any } n \in \mathbb{Z}, \quad (2.49)$$

$$\text{wt}(L(j)w) = \text{wt } w - j \quad \text{for } j = 0, \pm 1. \quad (2.50)$$

Proof Applying the $L(-1)$ -derivative property (2.40) to formula (2.29), with the operators acting on W , and extracting the coefficient of x^{-n-1} , we obtain:

$$[L(0), v_n] = (L(0)v)_n + (-n-1)v_n. \quad (2.51)$$

This can be written as

$$(L(0) - (\text{wt } v - n - 1))v_n = v_n L(0),$$

and so we have

$$(L(0) - (\text{wt } v + m - n - 1))v_n = v_n(L(0) - m)$$

for any $m \in \mathbb{C}$. Applying this repeatedly we get

$$(L(0) - (\text{wt } v + m - n - 1))^t v_n = v_n(L(0) - m)^t$$

for any $t \in \mathbb{N}$, $m \in \mathbb{C}$, and (2.49) follows.

For (2.50), since as operators acting on W we have

$$[L(0), L(j)] = -jL(j) \quad (2.52)$$

for $j = 0, \pm 1$, we get $(L(0) + j)L(j) = L(j)L(0)$ so that

$$(L(0) - m + j)L(j) = L(j)(L(0) - m)$$

for any $m \in \mathbb{C}$. Thus

$$(L(0) - m + j)^t L(j) = L(j)(L(0) - m)^t$$

for any $t \in \mathbb{N}$, $m \in \mathbb{C}$, and (2.50) follows. \square

Remark 2.20 From Proposition 2.19 we see that a generalized V -module W decomposes into submodules corresponding to the congruence classes of its weights modulo \mathbb{Z} : For $\mu \in \mathbb{C}/\mathbb{Z}$, let

$$W_{[\mu]} = \coprod_{\bar{n}=\mu} W_{[n]}, \quad (2.53)$$

where \bar{n} denotes the equivalence class of $n \in \mathbb{C}$ in \mathbb{C}/\mathbb{Z} . Then

$$W = \coprod_{\mu \in \mathbb{C}/\mathbb{Z}} W_{[\mu]} \quad (2.54)$$

and each $W_{[\mu]}$ is a V -submodule of W . Thus if a generalized module W is indecomposable (in particular, if it is irreducible), then all complex numbers n for which $W_{[n]} \neq 0$ are congruent modulo \mathbb{Z} to each other.

Remark 2.21 Let W be a generalized module for a Möbius (or conformal) vertex algebra V . We consider the “semisimple part” $L(0)_s \in \text{End } W$ of the operator $L(0)$:

$$L(0)_s w = n w \quad \text{for } w \in W_{[n]}, \quad n \in \mathbb{C}.$$

Then on W we have

$$[L(0)_s, v_n] = [L(0), v_n] \quad \text{for all } v \in V \text{ and } n \in \mathbb{Z}; \quad (2.55)$$

$$[L(0)_s, L(j)] = [L(0), L(j)] \quad \text{for } j = 0, \pm 1. \quad (2.56)$$

Indeed, for homogeneous elements $v \in V$ and $w \in W$, (2.49) and (2.51) imply that

$$\begin{aligned} [L(0)_s, v_n]w &= L(0)_s(v_n w) - v_n(L(0)_s w) \\ &= (\text{wt } v + \text{wt } w - n - 1)v_n w - (\text{wt } w)v_n w \\ &= (\text{wt } v)v_n w + (-n - 1)v_n w \\ &= (L(0)v)_n w + (-n - 1)v_n w \\ &= [L(0), v_n]w. \end{aligned}$$

Similarly, for any homogeneous element $w \in W$ and $j = 0, \pm 1$, (2.50) and (2.52) imply that

$$\begin{aligned} [L(0)_s, L(j)]w &= L(0)_s(L(j)w) - L(j)(L(0)_s w) \\ &= (\text{wt } w - j)L(j)w - (\text{wt } w)L(j)w \\ &= -jL(j)w \\ &= [L(0), L(j)]w. \end{aligned}$$

Thus the “locally nilpotent part” $L(0) - L(0)_s$ of $L(0)$ commutes with the action of V and of $\mathfrak{sl}(2)$ on W . In other words, $L(0) - L(0)_s$ is a V -homomorphism from W to itself.

Now suppose that $L(1)$ acts locally nilpotently on a Möbius (or conformal) vertex algebra V , that is, for any $v \in V$, there is $m \in \mathbb{N}$ such that $L(1)^m v = 0$. Then generalizing formula (3.20) in [HL5] (the case of ordinary modules for a vertex operator algebra), we define the *opposite vertex operator* on a generalized V -module (W, Y_W) associated to $v \in V$ by

$$Y_W^o(v, x) = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}), \quad (2.57)$$

that is, for $k \in \mathbb{Z}$ and $v \in V_{(k)}$,

$$\begin{aligned} Y_W^o(v, x) &= \sum_{n \in \mathbb{Z}} v_n^o x^{-n-1} \\ &= \sum_{n \in \mathbb{Z}} \left((-1)^k \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{-n-m-2+2k} \right) x^{-n-1}, \end{aligned} \quad (2.58)$$

as in [HL5]. (In the present work, we are replacing the symbol $*$ used in [HL5] for opposite vertex operators by the symbol o ; see also Subsection 5.1 below.) Here we are defining the component operators

$$v_n^o = (-1)^k \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{-n-m-2+2k} \quad (2.59)$$

for $v \in V_{(k)}$ and $n, k \in \mathbb{Z}$. Note that the $L(1)$ -local nilpotence ensures well-definedness here. Clearly, $v \mapsto Y_W^o(v, x)$ is a linear map $V \rightarrow (\text{End } W)[[x, x^{-1}]]$ such that $V \otimes W \rightarrow W((x^{-1}))$ ($v \otimes w \mapsto Y_W^o(v, x)w$).

By (2.59), (2.32) and (2.49), we see that for $n, k \in \mathbb{Z}$ and $v \in V_{(k)}$, the operator v_n^o is of generalized weight $n + 1 - k$ ($= n + 1 - \text{wt } v$), in the sense that

$$v_n^o W_{[m]} \subset W_{[m+n+1-k]} \quad \text{for any } m \in \mathbb{C}. \quad (2.60)$$

As mentioned in [HL5] (see (3.23) in [HL5]), the proof of Jacobi identity in Theorem 5.2.1 of [FHL] proves the following *opposite Jacobi identity* for Y_W^o in the case where V is a vertex operator algebra and W is a V -module:

$$\begin{aligned} &x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^o(v, x_2) Y_W^o(u, x_1) \\ &\quad - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W^o(u, x_1) Y_W^o(v, x_2) \\ &= x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W^o(Y(u, x_0)v, x_2) \end{aligned} \quad (2.61)$$

for $u, v \in V$, and taking Res_{x_0} gives us the *opposite commutator formula*. Similarly, the proof of the $L(-1)$ -derivative property in Theorem 5.2.1 of [FHL] proves the following *$L(-1)$ -derivative property* for Y_W^o in the same case:

$$\frac{d}{dx} Y_W^o(v, x) = Y_W^o(L(-1)v, x). \quad (2.62)$$

The same proofs carry over and prove the opposite Jacobi identity and the $L(-1)$ -derivative property for Y_W^o in the present case, where V is a Möbius (or conformal) vertex algebra with $L(1)$ acting locally nilpotently and where W is a generalized V -module. In the case in which V is a conformal vertex algebra, we have

$$Y_W^o(\omega, x) = Y_W(x^{-4}\omega, x^{-1}) = \sum_{n \in \mathbb{Z}} L(n)x^{n-2} \quad (2.63)$$

since $L(1)\omega = 0$.

For opposite vertex operators, we have the following analogues of (2.28)–(2.31) in the Möbius case:

Lemma 2.22 *For $v \in V$,*

$$[Y_W^o(v, x), L(1)] = Y_W^o(L(-1)v, x), \quad (2.64)$$

$$[Y_W^o(v, x), L(0)] = Y_W^o(L(0)v, x) + xY_W^o(L(-1)v, x), \quad (2.65)$$

$$[Y_W^o(v, x), L(-1)] = Y_W^o(L(1)v, x) + 2xY_W^o(L(0)v, x) + x^2Y_W^o(L(-1)v, x). \quad (2.66)$$

Equivalently,

$$\begin{aligned} [Y_W^o(v, x), L(-j)] &= \sum_{k=0}^{j+1} \binom{j+1}{k} x^k Y_W^o(L(j-k)v, x) \\ &= \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} Y_W^o(L(k-1)v, x) \end{aligned} \quad (2.67)$$

for $j = 0, \pm 1$.

Proof For $j = 0, \pm 1$, by definition and (2.31) we have

$$\begin{aligned} [Y_W^o(v, x), L(j)] &= -[L(j), Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})] \\ &= -\sum_{k=0}^{j+1} \binom{j+1}{k} x^{-k} Y_W(L(j-k)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}). \end{aligned} \quad (2.68)$$

By (5.2.14) in [FHL] and the fact that

$$x^{L(0)}L(j)x^{-L(0)} = x^{-j}L(j) \quad (2.69)$$

(easily proved by applying to a homogeneous vector),

$$\begin{aligned} &L(-1)e^{xL(1)}(-x^{-2})^{L(0)} \\ &= e^{xL(1)}L(-1)(-x^{-2})^{L(0)} - 2xe^{xL(1)}L(0)(-x^{-2})^{L(0)} + x^2e^{xL(1)}L(1)(-x^{-2})^{L(0)} \\ &= -x^2e^{xL(1)}(-x^{-2})^{L(0)}L(-1) - 2xe^{xL(1)}(-x^{-2})^{L(0)}L(0) - e^{xL(1)}(-x^{-2})^{L(0)}L(1) \\ &= -e^{xL(1)}(-x^{-2})^{L(0)}(x^2L(-1) + 2xL(0) + L(1)). \end{aligned} \quad (2.70)$$

We also have

$$\begin{aligned} L(1)e^{xL(1)}(-x^{-2})^{L(0)} &= e^{xL(1)}L(1)(-x^{-2})^{L(0)} \\ &= -x^{-2}e^{xL(1)}(-x^{-2})^{L(0)}L(1). \end{aligned} \quad (2.71)$$

By (2.70), (2.71), $L(0) = \frac{1}{2}[L(1), L(-1)]$ and $[L(1), L(0)] = L(1)$, we have

$$\begin{aligned} &L(0)e^{xL(1)}(-x^{-2})^{L(0)} \\ &= \frac{1}{2}L(1)L(-1)e^{xL(1)}(-x^{-2})^{L(0)} - \frac{1}{2}L(-1)L(1)e^{xL(1)}(-x^{-2})^{L(0)} \\ &= -\frac{1}{2}L(1)e^{xL(1)}(-x^{-2})^{L(0)}(x^2L(-1) + 2xL(0) + L(1)) \\ &\quad + \frac{1}{2}x^{-2}L(-1)e^{xL(1)}(-x^{-2})^{L(0)}L(1) \\ &= \frac{1}{2}x^{-2}e^{xL(1)}(-x^{-2})^{L(0)}L(1)(x^2L(-1) + 2xL(0) + L(1)) \\ &\quad - \frac{1}{2}x^{-2}e^{xL(1)}(-x^{-2})^{L(0)}(x^2L(-1) + 2xL(0) + L(1))L(1) \\ &= e^{xL(1)}(-x^{-2})^{L(0)}L(0) + x^{-1}e^{xL(1)}(-x^{-2})^{L(0)}L(1) \\ &= e^{xL(1)}(-x^{-2})^{L(0)}(L(0) + x^{-1}L(1)). \end{aligned} \quad (2.72)$$

Thus we obtain

$$\begin{aligned} &[Y_W^o(v, x), L(1)] \\ &= -\sum_{k=0}^2 \binom{2}{k} x^{-k} Y_W(L(1-k)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\ &= -Y_W(L(1)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) - 2x^{-1}Y_W(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\ &\quad - x^{-2}Y_W(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\ &= x^{-2}Y_W(e^{xL(1)}(-x^{-2})^{L(0)}L(1)v, x^{-1}) \\ &\quad - 2x^{-1}Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(L(0) + x^{-1}L(1))v, x^{-1}) \\ &\quad + x^{-2}Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(x^2L(-1) + 2xL(0) + L(1))v, x^{-1}) \\ &= Y_W(e^{xL(1)}(-x^{-2})^{L(0)}L(-1)v, x^{-1}) \\ &= Y_W^o(L(-1)v, x), \end{aligned}$$

$$\begin{aligned} &[Y_W^o(v, x), L(0)] \\ &= -\sum_{k=0}^1 \binom{1}{k} x^{-k} Y_W(L(-k)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\ &= -Y_W(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) - x^{-1}Y_W(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\ &= -Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(L(0) + x^{-1}L(1))v, x^{-1}) \end{aligned}$$

$$\begin{aligned}
& +x^{-1}Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(x^2L(-1)+2xL(0)+L(1))v, x^{-1}) \\
& = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(xL(-1)+L(0))v, x^{-1}) \\
& = Y_W^o(L(0)v, x) + xY_W^o(L(-1)v, x)
\end{aligned}$$

and

$$\begin{aligned}
[Y_W^o(v, x), L(-1)] & = -Y_W(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\
& = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(x^2L(-1)+2xL(0)+L(1))v, x^{-1}) \\
& = Y_W^o(L(1)v, x) + 2xY_W^o(L(0)v, x) + x^2Y_W^o(L(-1)v, x),
\end{aligned}$$

proving the lemma. \square

As in Section 5.2 of [FHL], we can define a V -action on W^* as follows:

$$\langle Y'(v, x)w', w \rangle = \langle w', Y_W^o(v, x)w \rangle \quad (2.73)$$

for $v \in V$, $w' \in W^*$ and $w \in W$; the correspondence $v \mapsto Y'(v, x)$ is a linear map from V to $(\text{End } W^*)[[x, x^{-1}]]$. Writing

$$Y'(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

($v_n \in \text{End } W^*$), we have

$$\langle v_n w', w \rangle = \langle w', v_n^o w \rangle \quad (2.74)$$

for $v \in V$, $w' \in W^*$ and $w \in W$. (Actually, in [FHL] this V -action was defined on a space smaller than W^* , but this definition holds without change on all of W^* .) In the case in which V is a conformal vertex algebra we define the operators $L'(n)$ ($n \in \mathbb{Z}$) by

$$Y'(\omega, x) = \sum_{n \in \mathbb{Z}} L'(n) x^{-n-2};$$

then, by extracting the coefficient of x^{-n-2} in (2.73) with $v = \omega$ and using the fact that $L(1)\omega = 0$ we have

$$\langle L'(n)w', w \rangle = \langle w', L(-n)w \rangle \quad \text{for } n \in \mathbb{Z} \quad (2.75)$$

(see (2.63)), as in Section 5.2 of [FHL]. In the case where V is only a Möbius vertex algebra, we define operators $L'(-1)$, $L'(0)$ and $L'(1)$ on W^* by formula (2.75) for $n = 0, \pm 1$. It follows from (2.50) that

$$L'(j)(W_{[m]})^* \subset (W_{[m-j]})^* \quad (2.76)$$

for $m \in \mathbb{C}$ and $j = 0, \pm 1$. By combining (2.74) with (2.60) we get

$$v_n(W_{[m]})^* \subset (W_{[m+k-n-1]})^* \quad (2.77)$$

for any $n, k \in \mathbb{Z}$, $v \in V_{(k)}$ and $m \in \mathbb{C}$.

We have just seen that the $L(1)$ -local nilpotence condition enables us to define a natural vertex operator action on the vector space dual of a generalized module for a Möbius (or

conformal) vertex algebra. This condition is satisfied by all vertex operator algebras, due to (2.32) and the grading restriction condition (2.24). However, the functor $W \mapsto W^*$ is certainly not involutive, and W^* is not in general a generalized module. In this work we will need certain module categories equipped with an involutive “contragredient functor” $W \mapsto W'$ which generalizes the contragredient functor for the category of modules for vertex operator algebras. For this purpose, we introduce the following:

Definition 2.23 Let A be an abelian group. A Möbius (or conformal) vertex algebra

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}$$

is said to be *strongly graded with respect to A* (or *strongly A -graded*, or just *strongly graded* if the abelian group A is understood) if there is a second gradation on V , by A ,

$$V = \coprod_{\alpha \in A} V^{(\alpha)},$$

such that the following conditions are satisfied: the two gradations are compatible, that is,

$$V^{(\alpha)} = \coprod_{n \in \mathbb{Z}} V_{(n)}^{(\alpha)} \quad (\text{where } V_{(n)}^{(\alpha)} = V_{(n)} \cap V^{(\alpha)}) \quad \text{for any } \alpha \in A;$$

for any $\alpha, \beta \in A$ and $n \in \mathbb{Z}$,

$$V_{(n)}^{(\alpha)} = 0 \quad \text{for } n \text{ sufficiently negative}; \quad (2.78)$$

$$\dim V_{(n)}^{(\alpha)} < \infty; \quad (2.79)$$

$$\mathbf{1} \in V_{(0)}^{(0)}; \quad (2.80)$$

$$v_l V^{(\beta)} \subset V^{(\alpha+\beta)} \quad \text{for any } v \in V^{(\alpha)}, l \in \mathbb{Z}; \quad (2.81)$$

and

$$L(j)V^{(\alpha)} \subset V^{(\alpha)} \quad \text{for } j = 0, \pm 1. \quad (2.82)$$

If V is in fact a conformal vertex algebra, we in addition require that

$$\omega \in V_{(2)}^{(0)}, \quad (2.83)$$

so that for all $j \in \mathbb{Z}$, (2.82) follows from (2.81).

Remark 2.24 Note that the notion of conformal vertex algebra strongly graded with respect to the trivial group is exactly the notion of vertex operator algebra. Also note that (2.32), (2.78) and (2.82) imply the local nilpotence of $L(1)$ acting on V , and hence we have the construction and properties of opposite vertex operators on a generalized module for a strongly graded Möbius (or conformal) vertex algebra.

For (generalized) modules for a strongly graded algebra we will also have a second grading by an abelian group, and it is natural to allow this group to be larger than the second grading group A for the algebra. (Note that this already occurs for the *first* grading group, which is \mathbb{Z} for algebras and \mathbb{C} for (generalized) modules.)

Definition 2.25 Let A be an abelian group and V a strongly A -graded Möbius (or conformal) vertex algebra. Let \tilde{A} be an abelian group containing A as a subgroup. A V -module (respectively, generalized V -module)

$$W = \coprod_{n \in \mathbb{C}} W_{(n)} \quad (\text{respectively, } W = \coprod_{n \in \mathbb{C}} W_{[n]})$$

is said to be *strongly graded with respect to \tilde{A}* (or *strongly \tilde{A} -graded*, or just *strongly graded*) if the abelian group \tilde{A} is understood) if there is a second gradation on W , by \tilde{A} ,

$$W = \coprod_{\beta \in \tilde{A}} W^{(\beta)}, \quad (2.84)$$

such that the following conditions are satisfied: the two gradations are compatible, that is, for any $\beta \in \tilde{A}$,

$$W^{(\beta)} = \coprod_{n \in \mathbb{C}} W_{(n)}^{(\beta)} \quad (\text{where } W_{(n)}^{(\beta)} = W_{(n)} \cap W^{(\beta)})$$

$$(\text{respectively, } W^{(\beta)} = \coprod_{n \in \mathbb{C}} W_{[n]}^{(\beta)} \quad (\text{where } W_{[n]}^{(\beta)} = W_{[n]} \cap W^{(\beta)});$$

for any $\alpha \in A$, $\beta \in \tilde{A}$ and $n \in \mathbb{C}$,

$$W_{(n+k)}^{(\beta)} = 0 \quad (\text{respectively, } W_{[n+k]}^{(\beta)} = 0) \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative}; \quad (2.85)$$

$$\dim W_{(n)}^{(\beta)} < \infty \quad (\text{respectively, } \dim W_{[n]}^{(\beta)} < \infty); \quad (2.86)$$

$$v_l W^{(\beta)} \subset W^{(\alpha+\beta)} \quad \text{for any } v \in V^{(\alpha)}, l \in \mathbb{Z}; \quad (2.87)$$

and

$$L(j)W^{(\beta)} \subset W^{(\beta)} \quad \text{for } j = 0, \pm 1. \quad (2.88)$$

(Note that if V is in fact a conformal vertex algebra, then for all $j \in \mathbb{Z}$, (2.88) follows from (2.83) and (2.87).)

Remark 2.26 A strongly A -graded conformal or Möbius vertex algebra is a strongly A -graded module for itself (and in particular, a strongly A -graded generalized module for itself).

Remark 2.27 Let V be a vertex operator algebra, viewed (equivalently) as a conformal vertex algebra strongly graded with respect to the trivial group (recall Remark 2.24). Then the V -modules that are strongly graded with respect to the trivial group (in the sense of

Definition 2.25) are exactly the (\mathbb{C} -graded) modules for V as a vertex operator algebra, with the grading restrictions as follows: For $n \in \mathbb{C}$,

$$W_{(n+k)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative} \quad (2.89)$$

and

$$\dim W_{(n)} < \infty. \quad (2.90)$$

Also, the generalized V -modules that are strongly graded with respect to the trivial group are exactly the generalized V -modules (in the sense of Definition 2.12) such that for $n \in \mathbb{C}$,

$$W_{[n+k]} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative} \quad (2.91)$$

and

$$\dim W_{[n]} < \infty. \quad (2.92)$$

Remark 2.28 In the strongly graded case, algebra and module homomorphisms are of course understood to preserve the grading by A or \tilde{A} .

Example 2.29 An important source of examples of strongly graded conformal vertex algebras and modules comes from the vertex algebras and modules associated with even lattices. Let L be an even lattice, i.e., a finite-rank free abelian group equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, not necessarily positive definite, such that $\langle a, a \rangle \in 2\mathbb{Z}$ for all $a \in L$. Then there is a natural structure of conformal vertex algebra on a certain vector space V_L ; see [B] and Chapter 8 of [FLM2]. If the form $\langle \cdot, \cdot \rangle$ on L is also positive definite, then V_L is a vertex operator algebra (that is, the grading restrictions hold). If L is not necessarily positive definite, then V_L is equipped with a natural second grading given by L itself, making V_L a strongly L -graded conformal vertex algebra in the sense of Definition 2.23. Any (rational) sublattice M of the “dual lattice” L° of L containing L gives rise to a strongly M -graded module for the strongly L -graded conformal vertex algebra (see Chapter 8 of [FLM2]; cf. [LL]).

Remark 2.30 As mentioned in Remark 2.24, strong gradedness for a Möbius (or conformal) vertex algebra V implies the local nilpotence of $L(1)$ acting on V . In fact, strong gradedness implies much more that will be important for us: From (2.78), (2.79), (2.81) and (2.82) (and (2.83) in the conformal vertex algebra case), it is clear that strong gradedness for V implies the following *local grading restriction condition* on V (see [H5]):

- (i) for any $m > 0$ and $v_{(1)}, \dots, v_{(m)} \in V$, there exists $r \in \mathbb{Z}$ such that the coefficient of each monomial in x_1, \dots, x_{m-1} in the formal series $Y(v_{(1)}, x_1) \cdots Y(v_{(m-1)}, x_{m-1})v_{(m)}$ lies in $\coprod_{n > r} V_{(n)}$;
- (ii) in the conformal vertex algebra case: for any element of the conformal vertex algebra V homogeneous with respect to the weight grading, the Virasoro-algebra submodule $M = \coprod_{n \in \mathbb{Z}} M_{(n)}$ (where $M_{(n)} = M \cap V_{(n)}$) of V generated by this element satisfies the following grading restriction conditions: $M_{(n)} = 0$ when n is sufficiently negative and $\dim M_{(n)} < \infty$ for $n \in \mathbb{Z}$

or

- (ii') in the Möbius vertex algebra case: for any element of the Möbius vertex algebra V homogeneous with respect to the weight grading, the $\mathfrak{sl}(2)$ -submodule $M = \coprod_{n \in \mathbb{Z}} M_{(n)}$ (where $M_{(n)} = M \cap V_{(n)}$) of V generated by this element satisfies the following grading restriction conditions: $M_{(n)} = 0$ when n is sufficiently negative and $\dim M_{(n)} < \infty$ for $n \in \mathbb{Z}$.

As was pointed out in [H5], Condition (i) above was first stated in [DL] (see formula (9.39), Proposition 9.17 and Theorem 12.33 in [DL]) for generalized vertex algebras and abelian intertwining algebras (certain generalizations of vertex algebras); it guarantees the convergence, rationality and commutativity properties of the matrix coefficients of products of more than two vertex operators. Conditions (i) and (ii) (or (ii')) together ensure that all the essential results involving the Virasoro operators and the geometry of vertex operator algebras in [H4] still hold for these algebras.

Remark 2.31 Similarly, from (2.85), (2.86), (2.87) and (2.88) (and (2.83) in the conformal vertex algebra case), it is clear that strong gradedness for (generalized) modules implies the following *local grading restriction condition* for a (generalized) module W for a strongly graded Möbius (or conformal) vertex algebra V :

- (i) for any $m > 0$, $v_{(1)}, \dots, v_{(m-1)} \in V$, $n \in \mathbb{C}$ and $w \in W_{[n]}$, there exists $r \in \mathbb{Z}$ such that the coefficient of each monomial in x_1, \dots, x_{m-1} in the formal series $Y(v_{(1)}, x_1) \cdots Y(v_{(m-1)}, x_{m-1})w$ lies in $\coprod_{k > r} W_{[n+k]}$;
- (ii) in the conformal vertex algebra case: for any $w \in W_{[n]}$ ($n \in \mathbb{C}$), the Virasoro-algebra submodule $M = \coprod_{k \in \mathbb{Z}} M_{[n+k]}$ (where $M_{[n+k]} = M \cap W_{[n+k]}$) of W generated by w satisfies the following grading restriction conditions: $M_{[n+k]} = 0$ when k is sufficiently negative and $\dim M_{[n+k]} < \infty$ for $k \in \mathbb{Z}$

or

- (ii') in the Möbius vertex algebra case: for any $w \in W_{[n]}$ ($n \in \mathbb{C}$), the $\mathfrak{sl}(2)$ -submodule $M = \coprod_{k \in \mathbb{Z}} M_{[n+k]}$ (where $M_{[n+k]} = M \cap W_{[n+k]}$) of W generated by w satisfies the following grading restriction conditions: $M_{[n+k]} = 0$ when k is sufficiently negative and $\dim M_{[n+k]} < \infty$ for $k \in \mathbb{Z}$.

Note that in the case of ordinary (as opposed to generalized) modules, all the generalized weight spaces such as $W_{[n]}$ mentioned here are ordinary weight spaces $W_{(n)}$.

With the strong gradedness condition on a (generalized) module, we can now define the corresponding notion of contragredient module. First we give:

Definition 2.32 Let $W = \coprod_{\beta \in \tilde{A}, n \in \mathbb{C}} W_{[n]}^{(\beta)}$ be a strongly \tilde{A} -graded generalized module for a strongly A -graded Möbius (or conformal) vertex algebra. For each $\beta \in \tilde{A}$ and $n \in \mathbb{C}$, let us

identify $(W_{[n]}^{(\beta)})^*$ with the subspace of W^* consisting of the linear functionals on W vanishing on each $W_{[m]}^{(\gamma)}$ with $\gamma \neq \beta$ or $m \neq n$ (cf. (2.48)). We define W' to be the $(\tilde{A} \times \mathbb{C})$ -graded vector subspace of W^* given by

$$W' = \coprod_{\beta \in \tilde{A}, n \in \mathbb{C}} (W')_{[n]}^{(\beta)}, \quad \text{where } (W')_{[n]}^{(\beta)} = (W_{[n]}^{(-\beta)})^*; \quad (2.93)$$

we also use the notations

$$(W')^{(\beta)} = \coprod_{n \in \mathbb{C}} (W_{[n]}^{(-\beta)})^* \subset (W^{(-\beta)})^* \subset W^* \quad (2.94)$$

(where $(W^{(\beta)})^*$ consists of the linear functionals on W vanishing on all $W^{(\gamma)}$ with $\gamma \neq \beta$) and

$$(W')_{[n]} = \coprod_{\beta \in \tilde{A}} (W_{[n]}^{(-\beta)})^* \subset (W_{[n]})^* \subset W^* \quad (2.95)$$

for the homogeneous subspaces of W' with respect to the \tilde{A} - and \mathbb{C} -grading, respectively (The reason for the minus signs here will become clear below.) We will still use the notation $\langle \cdot, \cdot \rangle_W$, or $\langle \cdot, \cdot \rangle$ when the underlying space is clear, for the canonical pairing between W' and $\overline{W} \subset \coprod_{\beta \in \tilde{A}, n \in \mathbb{C}} W_{[n]}^{(\beta)}$ (recall (2.47)).

Remark 2.33 In the case of ordinary rather than generalized modules, Definition 2.32 still applies, and all of the generalized weight subspaces $W_{[n]}$ of W are ordinary weight spaces $W_{(n)}$. In this case, we can write $(W')_{(n)}$ rather than $(W')_{[n]}$ for the corresponding subspace of W' .

Let W be a strongly graded (generalized) module for a strongly graded Möbius (or conformal) vertex algebra V . Recall that we have the action (2.73) of V on W^* and that (2.77) holds. Furthermore, (2.59), (2.74) and (2.87) imply for any $n, k \in \mathbb{Z}$, $\alpha \in A$, $\beta \in \tilde{A}$, $v \in V_{(k)}^{(\alpha)}$ and $m \in \mathbb{C}$,

$$v_n((W')_{[m]}^{(\beta)}) = v_n((W_{[m]}^{(-\beta)})^*) \subset (W_{[m+k-n-1]}^{(-\alpha-\beta)})^* = (W')_{[m+k-n-1]}^{(\alpha+\beta)}. \quad (2.96)$$

Thus v_n preserves W' for $v \in V$, $n \in \mathbb{Z}$. Similarly (in the Möbius case), (2.75), (2.76) and (2.88) imply that W' is stable under the operators $L'(-1)$, $L'(0)$ and $L'(1)$, and in fact

$$L'(j)(W')_{[n]}^{(\beta)} \subset (W')_{[n-j]}^{(\beta)}$$

for any $j = 0, \pm 1$, $\beta \in \tilde{A}$ and $n \in \mathbb{C}$. In the case of ordinary rather than generalized modules, the symbols $(W')_{[n]}^{(\beta)}$, etc., can be replaced by $(W')_{(n)}^{(\beta)}$, etc.

For any fixed $\beta \in \tilde{A}$ and $n \in \mathbb{C}$, by (2.43) and the finite-dimensionality (2.86) of $W_{[n]}^{(-\beta)}$, there exists $N \in \mathbb{N}$ such that $(L(0) - n)^N W_{[n]}^{(-\beta)} = 0$. But then for any $w' \in (W')_{[n]}^{(\beta)}$,

$$\langle (L'(0) - n)^N w', w \rangle = \langle w', (L(0) - n)^N w \rangle = 0 \quad (2.97)$$

for all $w \in W$. Thus $(L'(0) - n)^N w' = 0$. So (2.43) holds with W replaced by W' . In the case of ordinary modules, we of course take $N = 1$.

By (2.85) and (2.96) we have the lower truncation condition for the action Y' of V on W' :

$$\text{For any } v \in V \text{ and } w' \in W', v_n w' = 0 \text{ for } n \text{ sufficiently large.} \quad (2.98)$$

As a consequence, the Jacobi identity can now be formulated on W' . In fact, by the above, and using the same proofs as those of Theorems 5.2.1 and 5.3.1 in [FHL], together with Lemma 2.22, we obtain:

Theorem 2.34 *Let \tilde{A} be an abelian group containing A as a subgroup and V a strongly A -graded Möbius (or conformal) vertex algebra. Let (W, Y) be a strongly \tilde{A} -graded V -module (respectively, generalized V -module). Then the pair (W', Y') carries a strongly \tilde{A} -graded V -module (respectively, generalized V -module) structure, and $(W'', Y'') = (W, Y)$. \square*

Definition 2.35 The pair (W', Y') in Theorem 2.34 will be called the *contragredient module* of (W, Y) .

Let W_1 and W_2 be strongly \tilde{A} -graded (generalized) V -modules and let $f : W_1 \rightarrow W_2$ be a module homomorphism (which is of course understood to preserve both the \mathbb{C} -grading and the \tilde{A} -grading, and to preserve the action of $\mathfrak{sl}(2)$ in the Möbius case). Then by (2.74) and (2.75), the linear map $f' : W'_2 \rightarrow W'_1$ given by

$$\langle f'(w'_{(2)}), w_{(1)} \rangle = \langle w'_{(2)}, f(w_{(1)}) \rangle \quad (2.99)$$

for any $w_{(1)} \in W_1$ and $w'_{(2)} \in W'_2$ is well defined and is clearly a module homomorphism from W'_2 to W'_1 .

Notation 2.36 In this work we will be especially interested in the case where V is strongly graded, and we will be focusing on the category of all strongly graded modules, for which we will use the notation \mathcal{M}_{sg} , or the category of all strongly graded generalized modules, which we will call \mathcal{GM}_{sg} . From the above we see that in the strongly graded case we have contravariant functors

$$(\cdot)' : (W, Y) \mapsto (W', Y'),$$

the *contragredient functors*, from \mathcal{M}_{sg} to itself and from \mathcal{GM}_{sg} to itself. We also know that V itself is an object of \mathcal{M}_{sg} (and thus of \mathcal{GM}_{sg} as well); recall Remark 2.26. Our main objects of study will be certain full subcategories \mathcal{C} of \mathcal{M}_{sg} or \mathcal{GM}_{sg} that are closed under the contragredient functor and such that $V \in \text{ob } \mathcal{C}$.

Remark 2.37 In order to formulate certain results in this work, even in the case when our Möbius or conformal vertex algebra V is strongly graded we will in fact sometimes use the category whose objects are *all* the modules for V and whose morphisms are all the V -module homomorphisms, and also the category of *all* the generalized modules for V . (If V is conformal, then the category of all the V -modules is the same whether V is viewed as either conformal or Möbius, by Remark 2.14, and similarly for the category of all the generalized V -modules.) Note that in view of Remark 2.28, the categories \mathcal{M}_{sg} and \mathcal{GM}_{sg} are not full subcategories of these categories of *all* modules and generalized modules.

We now recall from [FLM2], [FHL], [DL] and [LL] the well-known principles that vertex operator algebras (which are exactly conformal vertex algebras strongly graded with respect to the trivial group; recall Remark 2.24) and their modules have important “rationality,” “commutativity” and “associativity” properties, and that these properties can in fact be used as axioms replacing the Jacobi identity in the definition of the notion of vertex operator algebra. (These principles in fact generalize to all vertex algebras, as in [LL].)

In the propositions below, $\mathbb{C}[x_1, x_2]_S$ is the ring of formal rational functions obtained by inverting (localizing with respect to) the products of (zero or more) elements of the set S of nonzero homogeneous linear polynomials in x_1 and x_2 . Also, ι_{12} (which might also be written as $\iota_{x_1 x_2}$) is the operation of expanding an element of $\mathbb{C}[x_1, x_2]_S$, that is, a polynomial in x_1 and x_2 divided by a product of homogeneous linear polynomials in x_1 and x_2 , as a formal series containing at most finitely many negative powers of x_2 (using binomial expansions for negative powers of linear polynomials involving both x_1 and x_2); similarly for ι_{21} and so on. (The distinction between rational functions and formal Laurent series is crucial.)

Let V be a vertex operator algebra. For W a (\mathbb{C} -graded) V -module (including possibly V itself), the space W' is just the “restricted dual space”

$$W' = \coprod W_{(n)}^*. \quad (2.100)$$

Proposition 2.38 *We have:*

(a) *(rationality of products) For $v, v_1, v_2 \in V$ and $v' \in V'$, the formal series*

$$\langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle, \quad (2.101)$$

which involves only finitely many negative powers of x_2 and only finitely many positive powers of x_1 , lies in the image of the map ι_{12} :

$$\langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle = \iota_{12}f(x_1, x_2), \quad (2.102)$$

where the (uniquely determined) element $f \in \mathbb{C}[x_1, x_2]_S$ is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t} \quad (2.103)$$

for some $g \in \mathbb{C}[x_1, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) *(commutativity) We also have*

$$\langle v', Y(v_2, x_2)Y(v_1, x_1)v \rangle = \iota_{21}f(x_1, x_2). \quad (2.104)$$

Proposition 2.39 *We have:*

(a) *(rationality of iterates) For $v, v_1, v_2 \in V$ and $v' \in V'$, the formal series*

$$\langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle, \quad (2.105)$$

which involves only finitely many negative powers of x_0 and only finitely many positive powers of x_2 , lies in the image of the map ι_{20} :

$$\langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle = \iota_{20}h(x_0, x_2), \quad (2.106)$$

where the (uniquely determined) element $h \in \mathbb{C}[x_0, x_2]_S$ is of the form

$$h(x_0, x_2) = \frac{k(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t} \quad (2.107)$$

for some $k \in \mathbb{C}[x_0, x_2]$ and $r, s, t \in \mathbb{Z}$.

- (b) The formal series $\langle v', Y(v_1, x_0 + x_2)Y(v_2, x_2)v \rangle$, which involves only finitely many negative powers of x_2 and only finitely many positive powers of x_0 , lies in the image of ι_{02} , and in fact

$$\langle v', Y(v_1, x_0 + x_2)Y(v_2, x_2)v \rangle = \iota_{02}h(x_0, x_2). \quad (2.108)$$

Proposition 2.40 (associativity) *We have the following equality of formal rational functions:*

$$\iota_{12}^{-1} \langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle = (\iota_{20}^{-1} \langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle) \Big|_{x_0=x_1-x_2}, \quad (2.109)$$

that is,

$$f(x_1, x_2) = h(x_1 - x_2, x_2).$$

Proposition 2.41 *In the presence of the other axioms for the notion of vertex operator algebra, the Jacobi identity follows from the rationality of products and iterates, and commutativity and associativity. In particular, in the definition of vertex operator algebra, the Jacobi identity may be replaced by these properties.*

The rationality, commutativity and associativity properties immediately imply the following result, in which the formal variables x_1 and x_2 are specialized to nonzero complex numbers in suitable domains:

Corollary 2.42 *The formal series obtained by specializing x_1 and x_2 to (nonzero) complex numbers z_1 and z_2 , respectively, in (2.101) converges to a rational function of z_1 and z_2 in the domain*

$$|z_1| > |z_2| > 0 \quad (2.110)$$

and the analogous formal series obtained by specializing x_1 and x_2 to z_1 and z_2 , respectively, in (2.104) converges to the same rational function of z_1 and z_2 in the (disjoint) domain

$$|z_2| > |z_1| > 0. \quad (2.111)$$

Moreover, the formal series obtained by specializing x_0 and x_2 to $z_1 - z_2$ and z_2 , respectively, in (2.105) converges to this same rational function of z_1 and z_2 in the domain

$$|z_2| > |z_1 - z_2| > 0. \quad (2.112)$$

In particular, in the common domain

$$|z_1| > |z_2| > |z_1 - z_2| > 0, \quad (2.113)$$

we have the equality

$$\langle v', Y(v_1, z_1)Y(v_2, z_2)v \rangle = \langle v', Y(Y(v_1, z_1 - z_2)v_2, z_2)v \rangle \quad (2.114)$$

of rational functions of z_1 and z_2 .

Remark 2.43 These last five results also hold for modules for a vertex operator algebra V ; in the statements, one replaces the vectors v and v' by elements w and w' of a V -module W and its restricted dual W' , respectively, and Proposition 2.41 becomes: Given a vertex operator algebra V , in the presence of the other axioms for the notion of V -module, the Jacobi identity follows from the rationality of products and iterates, and commutativity and associativity. In particular, in the definition of V -module, the Jacobi identity may be replaced by these properties.

For either vertex operator algebras or modules, it is sometimes convenient to express the equalities of rational functions in Corollary 2.42 informally as follows:

$$Y(v_1, z_1)Y(v_2, z_2) \sim Y(v_2, z_2)Y(v_1, z_1) \quad (2.115)$$

and

$$Y(v_1, z_1)Y(v_2, z_2) \sim Y(Y(v_1, z_1 - z_2)v_2, z_2), \quad (2.116)$$

meaning that these expressions, defined in the domains indicated in Corollary 2.42 when the “matrix coefficients” of these expressions are taken as in this corollary, agree as operator-valued rational functions, up to analytic continuation.

Remark 2.44 Formulas (2.115) and (2.116) (or more precisely, (2.114)), express the meromorphic, or single-valued, version of “duality,” in the language of conformal field theory. Formulas (2.116) (and (2.114)) express the existence and associativity of the single-valued, or meromorphic, operator product expansion. This is the statement that the product of two (vertex) operators can be expanded as a (suitable, convergent) infinite sum of vertex operators, and that this sum can be expressed in the form of an iterate of vertex operators, parametrized by the complex numbers $z_1 - z_2$ and z_2 , in the format indicated; the infinite sum comes from expanding $Y(v_1, z_1 - z_2)v_2, z_2$ in powers of $z_1 - z_2$. A central goal of this work is to generalize (2.115) and (2.116), or more precisely, (2.114), to logarithmic intertwining operators in place of the operators $Y(\cdot, z)$. This will give the existence and also the associativity of the general, nonmeromorphic operator product expansion. This was done in the non-logarithmic setting in [HL5]–[HL7] and [H1]. In the next section, we shall develop the concept of logarithmic intertwining operator.

3 Logarithmic intertwining operators

In this section we study the notion of “logarithmic intertwining operator” introduced in [Mi]. For this, we will need to discuss spaces of formal series in powers of both x and “ $\log x$ ”, a new formal variable, with coefficients in certain vector spaces. We establish certain properties, some of them quite subtle, of the formal derivative operator d/dx acting on such spaces. Then, following [Mi] with a slight variant (see Remark 3.25), we introduce the notion of logarithmic intertwining operator. These are the appropriate replacements of ordinary intertwining operators when $L(0)$ -semisimplicity is relaxed. In the strongly graded setting, it will be natural to consider the associated “grading-compatible” logarithmic intertwining operators. We work out some important principles and formulas concerning logarithmic intertwining operators, certain of which turn out to be the same as in the ordinary intertwining operator case. Some of these results require proofs that are quite delicate.

Recall the notation $\mathcal{W}\{x\}$ (2.1) for the space of formal series in a formal variable x with coefficients in a vector space \mathcal{W} , with arbitrary complex powers of x .

From now on we will sometimes need and use new independent (commuting) formal variables called $\log x, \log y, \log x_1, \log x_2, \dots$, etc. We will work with formal series in such formal variables together with the “usual” formal variables x, y, x_1, x_2, \dots , etc., with coefficients in certain vector spaces, and the powers of the monomials in *all* the variables can be arbitrary complex numbers. (Later we will restrict our attention to only nonnegative integral powers of the “log” variables.)

Given a formal variable x , we will use the notation $\frac{d}{dx}$ to denote the linear map on $\mathcal{W}\{x, \log x\}$, for any vector space \mathcal{W} not involving x , defined (and indeed well defined) by the (expected) formula

$$\begin{aligned} \frac{d}{dx} \left(\sum_{m,n \in \mathbb{C}} w_{n,m} x^n (\log x)^m \right) \\ = \sum_{m,n \in \mathbb{C}} ((n+1)w_{n+1,m} + (m+1)w_{n+1,m+1}) x^n (\log x)^m \\ \left(= \sum_{m,n \in \mathbb{C}} n w_{n,m} x^{n-1} (\log x)^m + \sum_{m,n \in \mathbb{C}} m w_{n,m} x^{n-1} (\log x)^{m-1} \right) \end{aligned} \quad (3.1)$$

where $w_{n,m} \in \mathcal{W}$ for all $m, n \in \mathbb{C}$. We will also use the same notation for the restriction of $\frac{d}{dx}$ to any subspace of $\mathcal{W}\{x, \log x\}$ which is closed under $\frac{d}{dx}$, e.g., $\mathcal{W}\{x\}[[\log x]]$ or $\mathbb{C}[x, x^{-1}, \log x]$. Clearly, $\frac{d}{dx}$ acting on $\mathcal{W}\{x\}$ coincides with the usual formal derivative.

Remark 3.1 Let f, g and f_i, i in some index set I , all be formal series of the form

$$\sum_{m,n \in \mathbb{C}} w_{n,m} x^n (\log x)^m \in \mathcal{W}\{x, \log x\}, \quad w_{n,m} \in \mathcal{W}. \quad (3.2)$$

One checks the following straightforwardly: Suppose that the sum of $f_i, i \in I$, exists (in the obvious sense). Then the sum of the $\frac{d}{dx} f_i, i \in I$, also exists and is equal to $\frac{d}{dx} \sum_{i \in I} f_i$. More

generally, for any $T = p(x)\frac{d}{dx}$, $p(x) \in \mathbb{C}[x, x^{-1}]$, the sum of Tf_i , $i \in I$, exists and is equal to $T \sum_{i \in I} f_i$. Thus the sum of $e^{yT} f_i$, $i \in I$, exists and is equal to $e^{yT} \sum_{i \in I} f_i$ (e^{yT} being the formal exponential series, as usual). Suppose that \mathcal{W} is an (associative) algebra or that the coefficients of either f or g are complex numbers. If the product of f and g exists, then the product of $\frac{d}{dx}f$ and g and the product of f and $\frac{d}{dx}g$ both exist, and $\frac{d}{dx}(fg) = (\frac{d}{dx}f)g + f(\frac{d}{dx}g)$. Furthermore, for any T as before, the product of Tf and g and the product of f and Tg both exist, and $T(fg) = (Tf)g + f(Tg)$. In addition, the product of $e^{yT}f$ and $e^{yT}g$ exists and is equal to $e^{yT}(fg)$, just as in formulas (8.2.6)–(8.2.10) of [FLM2]. The point here, of course, is just the formal derivation property of $\frac{d}{dx}$, except that sums and products of expressions do not exist in general.

Remark 3.2 Note that the “equality” $x = e^{\log x}$ does not hold, since the left-hand side is a formal variable, while the right-hand side is a formal series in another formal variable. In fact, this formula should not be assumed, since, for example, the formal delta function $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$ would not exist in the sense of formal calculus, if x were allowed to be replaced by the formal series $e^{\log x}$. By contrast, note that the equality

$$\log e^x = x \tag{3.3}$$

does indeed hold. This is because the formal series e^x is of the form $1 + X$ where X involves only positive integral powers of x and in (3.3), “log” refers to the usual formal logarithmic series

$$\log(1 + X) = \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} X^i, \tag{3.4}$$

not to the “log” of a formal variable. We will use the symbol “log” in both ways, and the meaning will be clear in context.

We will typically use notations of the form $f(x)$, instead of $f(x, \log x)$, to denote elements of $\mathcal{W}\{x, \log x\}$ for some vector space \mathcal{W} as above. For this reason, we need to interpret carefully the meaning of symbols such as $f(x + y)$, or more generally, symbols obtained by replacing x in $f(x)$ by something other than just a single formal variable (since $\log x$ is a formal variable and not the image of some operator acting on x). For the three main types of cases that will be encountered in this work, we use the following notational conventions; the existence of the expressions will be justified in Remark 3.4:

Notation 3.3 For formal variables x and y , and $f(x)$ of the form (3.2), we define

$$\begin{aligned} f(x + y) &= \sum_{m, n \in \mathbb{C}} w_{n, m} (x + y)^n \left(\log x + \log \left(1 + \frac{y}{x} \right) \right)^m \\ &= \sum_{m, n \in \mathbb{C}} w_{n, m} (x + y)^n \left(\log x + \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left(\frac{y}{x} \right)^i \right)^m \end{aligned} \tag{3.5}$$

(recall (3.4)); in the right-hand side, $(\log x + \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} (\frac{y}{x})^i)^m$, according to the binomial expansion convention, is to be expanded in nonnegative integral powers of the second summand $\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} (\frac{y}{x})^i$, so the right-hand side of (3.5) is equal to

$$\sum_{m,n \in \mathbb{C}} w_{n,m} (x+y)^n \sum_{j \in \mathbb{N}} \binom{m}{j} (\log x)^{m-j} \left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left(\frac{y}{x} \right)^i \right)^j \quad (3.6)$$

when expanded one step further. Also define

$$f(xe^y) = \sum_{m,n \in \mathbb{C}} w_{n,m} x^n e^{ny} (\log x + y)^m \quad (3.7)$$

and

$$f(xy) = \sum_{m,n \in \mathbb{C}} w_{n,m} x^n y^n (\log x + \log y)^m, \quad (3.8)$$

where the binomial expansion convention is again of course being used.

Remark 3.4 The existence of the right-hand side of (3.5), or (3.6), can be seen by writing $(x+y)^n$ as $x^n (1 + \frac{y}{x})^n$ and observing that

$$\left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left(\frac{y}{x} \right)^i \right)^j \in \left(\frac{y}{x} \right)^j \mathbb{C} \left[\left[\frac{y}{x} \right] \right].$$

The existence of the right-hand sides of (3.7) and of (3.8) is clear. Furthermore, both $f(x+y)$ and $f(xe^y)$ lie in $\mathcal{W}\{x, \log x\}[[y]]$, while $f(xy)$ lies in $\mathcal{W}\{xy, \log x\}[[\log y]]$. One might expect that $f(x+y)$ can be written as $e^{y \frac{d}{dx}} f(x)$, and $f(xe^y)$ as $e^{yx \frac{d}{dx}} f(x)$ (cf. Section 8.3 of [FLM2]), but these formulas must be verified (see Theorem 3.6 below).

Remark 3.5 It is clear that when there is no $\log x$ involved in $f(x)$, the expression $f(x+y)$ (respectively, $f(xe^y)$, $f(xy)$) coincides with the usual formal operation of substitution of $x+y$ (respectively, xe^y , xy) for x in $f(x)$. In general, it is straightforward to check that if the sum of $f_i(x)$, $i \in I$, exists and is equal to $f(x)$, then the sum of $f_i(x+y)$, $i \in I$ (respectively, $f_i(xe^y)$, $i \in I$, $f_i(xy)$, $i \in I$), also exists and is equal to $f(x+y)$ (respectively, $f(xe^y)$, $f(xy)$). Also, suppose that \mathcal{W} is an (associative) algebra or that the coefficients of either f or g are complex numbers. If the product of $f(x)$ and $g(x)$ exists, then the product of $f(x+y)$ and $g(x+y)$ (respectively, $f(xe^y)$ and $g(xe^y)$, $f(xy)$ and $g(xy)$) also exists and is equal to $(fg)(x+y)$ (respectively, $(fg)(xe^y)$, $(fg)(xy)$).

Note that by (3.5),

$$\log(x+y) = \log x + \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left(\frac{y}{x} \right)^i = e^{y \frac{d}{dx}} \log x. \quad (3.9)$$

The next result includes a generalization of this to arbitrary elements of $\mathcal{W}\{x, \log x\}$. Formula (3.10) is a formal “Taylor theorem” for logarithmic formal series. In the case of non-logarithmic formal series, this principle is used extensively in vertex operator algebra theory; recall (2.2) above and see Proposition 8.3.1 of [FLM2] for the proof in the generality of formal series with arbitrary complex powers of the formal variable. In the logarithmic case below, a much more elaborate proof is required than in the non-logarithmic case. The other formula in Theorem 3.6, formula (3.11), is easier to prove. It too is important (in the non-logarithmic case) in vertex operator algebra theory; again see Proposition 8.3.1 of [FLM2].

Theorem 3.6 *For $f(x)$ as in (3.2), we have*

$$e^{y \frac{d}{dx}} f(x) = f(x + y) \quad (3.10)$$

(“Taylor’s theorem” for logarithmic formal series) and

$$e^{yx \frac{d}{dx}} f(x) = f(xe^y). \quad (3.11)$$

Proof By Remarks 3.1 and 3.5 (or 3.4), we need only prove these equalities for $f(x) = x^n$ and $f(x) = (\log x)^m$, $m, n \in \mathbb{C}$. The case $f(x) = x^n$ easily follows from the direct expansion of the two sides of (3.10) and of (3.11) (see Proposition 8.3.1 in [FLM2]). Now assume that $f(x) = (\log x)^m$, $m \in \mathbb{C}$.

Formula (3.11) is easier, so we prove it first. By (3.1) we have

$$x \frac{d}{dx} (\log x)^m = m (\log x)^{m-1},$$

so that for $k \in \mathbb{N}$,

$$\left(x \frac{d}{dx}\right)^k (\log x)^m = m(m-1) \cdots (m-k+1) (\log x)^{m-k} = k! \binom{m}{k} (\log x)^{m-k}.$$

Thus

$$e^{yx \frac{d}{dx}} (\log x)^m = \sum_{k \in \mathbb{N}} \frac{y^k}{k!} \left(x \frac{d}{dx}\right)^k (\log x)^m = \sum_{k \in \mathbb{N}} \binom{m}{k} y^k (\log x)^{m-k} = (\log x + y)^m,$$

as we want.

For (3.10), we shall give two proofs — an analytic proof and an algebraic proof. First, consider the analytic function $(\log z)^m = e^{m \log \log z}$ over, say, $|z-3| < 1$ in the complex plane. In this proof we take the branch of $\log z$ so that

$$-\pi < \operatorname{Im}(\log z) \leq \pi. \quad (3.12)$$

Then by analyticity, for any z in this domain, when $|z_1|$ is small enough the Taylor series expansion $e^{z_1 \frac{d}{dz}} (\log z)^m$ converges absolutely to $(\log(z + z_1))^m$. That is,

$$(\log(z + z_1))^m = e^{z_1 \frac{d}{dz}} (\log z)^m = e^{\frac{z_1}{z} z \frac{d}{dz}} (\log z)^m. \quad (3.13)$$

Observe that as a formal series, the right-hand side of (3.13) is in the space $(\log z)^m \mathbb{C}[(\log z)^{-1}][[z_1/z]]$.

On the other hand, by the choice of domain and the branch of \log we have

$$\log(z + z_1) = \log z + \log(1 + z_1/z)$$

and

$$|\log z| > \log 2 > |\log(1 + z_1/z)|$$

when $|z_1|$ is small enough. So when $|z_1|$ is small enough we have

$$\begin{aligned} (\log(z + z_1))^m &= (\log z + \log(1 + z_1/z))^m \\ &= \sum_{j \in \mathbb{N}} \binom{m}{j} (\log z)^{m-j} \left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left(\frac{z_1}{z} \right)^i \right)^j. \end{aligned} \quad (3.14)$$

Since as formal series, the right-hand sides of (3.13) and (3.14) are both in the space $(\log z)^m \mathbb{C}[(\log z)^{-1}][[z_1/z]]$, and both converge to the same analytic function $(\log(z + z_1))^m$ in the above domain, by setting $z_1 = 0$ in these two functions and their derivatives with respect to z_1 we see that their corresponding coefficients of powers of z_1/z and further, of all monomials in $\log z$ and z_1/z must be the same. Hence we can replace z and z_1 by formal variables x and y , respectively, and obtain (3.10) for $f(x) = (\log x)^m$.

An algebraic proof of (3.10) (for $(\log x)^m$) can be given as follows: Since

$$\frac{d}{dx}(\log x)^m = mx^{-1}(\log x)^{m-1}$$

and higher derivatives involve derivatives of products of powers of x and powers of $\log x$, let us first compute $(d/dx)^k(x^n(\log x)^m)$ directly for all $m, n \in \mathbb{C}$ and $k \in \mathbb{N}$. Define linear maps T_0 and T_1 on $\mathbb{C}\{x, \log x\}$ by setting

$$T_0 x^n (\log x)^m = nx^{n-1} (\log x)^m \quad \text{and} \quad T_1 x^n (\log x)^m = mx^{n-1} (\log x)^{m-1},$$

respectively, and extending to all of $\mathbb{C}\{x, \log x\}$ by formal linearity. Then the formula

$$\frac{d}{dx} x^n (\log x)^m = nx^{n-1} (\log x)^m + mx^{n-1} (\log x)^{m-1}$$

(extended to $\mathbb{C}\{x, \log x\}$) can be written as

$$\frac{d}{dx} = T_0 + T_1$$

on $\mathbb{C}\{x, \log x\}$. So for $k \geq 1$, on $\mathbb{C}\{x, \log x\}$,

$$\begin{aligned} \left(\frac{d}{dx} \right)^k &= \sum_{(i_1, \dots, i_k) \in \{0,1\}^k} T_{i_1} \cdots T_{i_k} \\ &= \sum_{j=0}^{k-1} \sum_{0 \leq t_1 < t_2 < \dots < t_{k-j} < k} T_1^{k-t_{k-j}-1} T_0 T_1^{t_{k-j}-t_{k-j-1}-1} T_0 \cdots T_0 T_1^{t_2-t_1-1} T_0 T_1^{t_1} + T_1^k, \end{aligned}$$

where j gives the number of T_1 's in the product $T_{i_1} \cdots T_{i_k}$; there are $k - j$ T_0 's, which are in the following positions, reading from the right: $t_1 + 1, t_2 + 1, \dots, t_{k-j} + 1$. That is, for $k \geq 1$,

$$\begin{aligned} \left(\frac{d}{dx}\right)^k x^n (\log x)^m &= \sum_{j=0}^k m(m-1) \cdots (m-j+1) \cdot \\ &\cdot \left(\sum_{0 \leq t_1 < t_2 < \cdots < t_{k-j} < k} (n-t_1)(n-t_2) \cdots (n-t_{k-j}) \right) x^{n-k} (\log x)^{m-j}, \end{aligned}$$

where it is understood that if $j = k$, then the latter sum (in parentheses) is 1. In this formula, setting $n = 0$, multiplying by $y^k/k!$, and then summing over $k \in \mathbb{N}$, we get (noting that $t_1 = 0$ contributes 0)

$$\begin{aligned} e^{y \frac{d}{dx}} (\log x)^m &= \sum_{k \in \mathbb{N}} \left(\frac{y}{x}\right)^k \sum_{j=0}^k \binom{m}{j} (\log x)^{m-j} \frac{j!}{k!} \cdot \\ &\cdot \left(\sum_{0 < t_1 < t_2 < \cdots < t_{k-j} < k} (-t_1)(-t_2) \cdots (-t_{k-j}) \right). \end{aligned} \quad (3.15)$$

So (3.10) for $f(x) = (\log x)^m$ is equivalent to equating the right-hand side of (3.15) to

$$\begin{aligned} \left(\log x + \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left(\frac{y}{x}\right)^i \right)^m &= \\ &= \sum_{j \in \mathbb{N}} \binom{m}{j} (\log x)^{m-j} \left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left(\frac{y}{x}\right)^i \right)^j \\ &= \sum_{j \in \mathbb{N}} \binom{m}{j} (\log x)^{m-j} \sum_{k \geq j} \left(\sum_{\substack{i_1 + \cdots + i_j = k \\ 1 \leq i_1, \dots, i_j \leq k}} \frac{(-1)^{k-j}}{i_1 i_2 \cdots i_j} \right) \left(\frac{y}{x}\right)^k \\ &= \sum_{k \in \mathbb{N}} \left(\frac{y}{x}\right)^k \sum_{j=0}^k \binom{m}{j} (\log x)^{m-j} \left(\sum_{\substack{i_1 + \cdots + i_j = k \\ 1 \leq i_1, \dots, i_j \leq k}} \frac{(-1)^{k-j}}{i_1 i_2 \cdots i_j} \right). \end{aligned} \quad (3.16)$$

Comparing the right-hand sides of (3.15) and (3.16) we see that it is equivalent to proving the combinatorial identity

$$\frac{j!}{k!} \sum_{0 < t_1 < t_2 < \cdots < t_{k-j} < k} t_1 t_2 \cdots t_{k-j} = \sum_{\substack{i_1 + \cdots + i_j = k \\ 1 \leq i_1, \dots, i_j \leq k}} \frac{1}{i_1 i_2 \cdots i_j} \quad (3.17)$$

for all $k \in \mathbb{N}$ and $j = 0, \dots, k$. Note that there is no m involved here. But for m a nonnegative integer, (3.10) for $f(x) = (\log x)^m$ follows from

$$e^{y \frac{d}{dx}} (\log x)^m = (e^{y \frac{d}{dx}} \log x)^m$$

(recall Remark 3.1) and

$$e^{y \frac{d}{dx}} \log x = \log x + \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \left(\frac{y}{x}\right)^i$$

(recall (3.9)). Thus the expressions in (3.15) and (3.16) are equal for any such m . Equating coefficients and choosing $m \geq j$ gives us (3.17). Therefore (3.10) also holds for any $m \in \mathbb{C}$. \square

Remark 3.7 Note that in both the left- and right-hand sides of (3.10) or (3.11), y can be replaced by a suitable formal series, for example, by an element of $y\mathbb{C}[x][[y]]$, and Theorem 3.6 in fact still holds if y is replaced by an arbitrary formal series in $y\mathbb{C}[x][[y]]$. We will exploit this later.

Remark 3.8 Here is an amusing sidelight: When we were writing up the proof above, one of us (L.Z.) happened to pick up the then-current issue of the American Mathematical Monthly and happened to notice the following problem from the Problems and Solutions section, proposed by D. Lubell [Lu]:

Let N and j be positive integers, and let $S = \{(w_1, \dots, w_j) \in \mathbb{Z}_+^j \mid 0 < w_1 + \dots + w_j \leq N\}$ and $T = \{(w_1, \dots, w_j) \in \mathbb{Z}_+^j \mid w_1, \dots, w_j \text{ are distinct and bounded by } N\}$. Show that

$$\sum_S \frac{1}{w_1 \cdots w_j} = \sum_T \frac{1}{w_1 \cdots w_j}.$$

But this follows immediately from (3.17) (which is in fact a refinement), since the left-hand side of (3.17) is equal to

$$j! \sum_{1 \leq w_1 < w_2 < \dots < w_{j-1} \leq k-1} \frac{1}{w_1 w_2 \cdots w_{j-1} k} = \sum_{T_k} \frac{1}{w_1 w_2 \cdots w_j}$$

where

$$T_k = \{(w_1, \dots, w_j) \in \{1, 2, \dots, k\}^j \mid w_i \text{ distinct, with maximum exactly } k\},$$

the right-hand side is

$$\sum_{S_k} \frac{1}{w_1 w_2 \cdots w_j}$$

where

$$S_k = \{(w_1, \dots, w_j) \in \{1, 2, \dots, k\}^j \mid w_1 + \dots + w_j = k\},$$

and one has $S = \coprod_{k=1}^N S_k$ and $T = \coprod_{k=1}^N T_k$.

When we define the notion of logarithmic intertwining operator below, we will impose a condition requiring certain formal series to lie in spaces of the type $\mathcal{W}[\log x]\{x\}$ (so that for each power of x , possibly complex, we have a *polynomial* in $\log x$), partly because such results as the following (which is expected) will indeed hold in our formal setup when the powers of the formal variables are restricted in this way (cf. Remark 3.10 below).

Lemma 3.9 *Let $a \in \mathbb{C}$ and $m \in \mathbb{Z}_+$. If $f(x) \in \mathcal{W}[\log x]\{x\}$ (\mathcal{W} any vector space not involving x or $\log x$) satisfies the formal differential equation*

$$\left(x \frac{d}{dx} - a\right)^m f(x) = 0, \quad (3.18)$$

then $f(x) \in \mathcal{W}x^a \oplus \mathcal{W}x^a \log x \oplus \cdots \oplus \mathcal{W}x^a (\log x)^{m-1}$, and furthermore, if m is the smallest integer so that (3.18) is satisfied, then the coefficient of $x^a (\log x)^{m-1}$ in $f(x)$ is nonzero.

Proof For any $f(x) = \sum_{n,k} w_{n,k} x^n (\log x)^k \in \mathcal{W}\{x, \log x\}$,

$$\begin{aligned} x \frac{d}{dx} f(x) &= \sum_{n,k} n w_{n,k} x^n (\log x)^k + \sum_{n,k} k w_{n,k} x^n (\log x)^{k-1} \\ &= \sum_{n,k} (n w_{n,k} + (k+1) w_{n,k+1}) x^n (\log x)^k. \end{aligned}$$

Thus for any $a \in \mathbb{C}$,

$$\left(x \frac{d}{dx} - a\right) f(x) = \sum_{n,k} ((n-a) w_{n,k} + (k+1) w_{n,k+1}) x^n (\log x)^k. \quad (3.19)$$

Now suppose that $f(x)$ lies in $\mathcal{W}[\log x]\{x\}$. Let us prove the assertion of the lemma by induction on m .

If $m = 1$, by (3.19) we see that $(x \frac{d}{dx} - a) f(x) = 0$ means that

$$(n-a) w_{n,k} + (k+1) w_{n,k+1} = 0 \quad \text{for any } n \in \mathbb{C}, k \in \mathbb{Z}. \quad (3.20)$$

Fix n . If $w_{n,k} \neq 0$ for some k , let k_n be the smallest nonnegative integer such that $w_{n,k} = 0$ for any $k > k_n$ (such a k_n exists because $f(x) \in \mathcal{W}[\log x]\{x\}$). Then

$$(n-a) w_{n,k_n} = -(k_n+1) w_{n,k_n+1} = 0.$$

But $w_{n,k_n} \neq 0$ by the choice of k_n , so we must have $n = a$. Now (3.20) becomes $(k+1) w_{a,k+1} = 0$ for any $k \in \mathbb{Z}$, so that $w_{a,k} = 0$ unless $k = 0$. Thus $f(x) = w_{a,0} x^a$. If in addition $m = 1$ is the smallest integer such that (3.18) holds, then $f(x) \neq 0$. So $w_{a,0}$, the coefficient of x^a , is not zero.

Suppose the statement is true for m . Then for the case $m+1$, since

$$0 = \left(x \frac{d}{dx} - a\right)^{m+1} f(x) = \left(x \frac{d}{dx} - a\right)^m \left(x \frac{d}{dx} - a\right) f(x) \quad (3.21)$$

implies that

$$\left(x \frac{d}{dx} - a\right) f(x) = \bar{w}_0 x^a + \bar{w}_1 x^a \log x + \cdots + \bar{w}_{m-1} x^a (\log x)^{m-1} \quad (3.22)$$

for some $\bar{w}_0, \bar{w}_1, \dots, \bar{w}_{m-1} \in \mathcal{W}$, by (3.19) we get

$$\begin{aligned} (n-a)w_{n,j} + (j+1)w_{n,j+1} &= 0 && \text{for any } n \neq a \text{ and any } j \in \mathbb{Z} \\ (j+1)w_{a,j+1} &= \bar{w}_j && \text{for any } j \in \{0, 1, \dots, m-1\} \\ (j+1)w_{a,j+1} &= 0 && \text{for any } j \notin \{0, 1, \dots, m-1\} \end{aligned}$$

By the same argument as above we get $w_{n,j} = 0$ for any $n \neq a$ and any j . So

$$f(x) = w_{a,0} x^a + \bar{w}_0 x^a \log x + \frac{\bar{w}_1}{2} x^a (\log x)^2 + \cdots + \frac{\bar{w}_{m-1}}{m} x^a (\log x)^m,$$

as we want. If in addition $m+1$ is the smallest integer so that (3.21) is satisfied, then by the induction assumption, \bar{w}_{m-1} in (3.22) is not zero. So the coefficient in $f(x)$ of $x^a (\log x)^m$, \bar{w}_{m-1}/m , is not zero, as we want. \square

Remark 3.10 Note that there are solutions of the equation (3.18) outside $\mathcal{W}[\log x]\{x\}$, for example, $f(x) = wx^b e^{(a-b)\log x} \in x^b \mathcal{W}[[\log x]]$ for any complex number $b \neq a$ and any $0 \neq w \in \mathcal{W}$.

Following [Mi], with a slight generalization (see Remark 3.25), we now introduce the notion of logarithmic intertwining operator, together with the notion of “grading-compatible logarithmic intertwining operator,” adapted to the strongly graded case. We will later see that the axioms in these definitions correspond exactly to those in the notion of certain “intertwining maps” (see Definition 4.2 below).

Definition 3.11 Let (W_1, Y_1) , (W_2, Y_2) and (W_3, Y_3) be generalized modules for a Möbius (or conformal) vertex algebra V . A *logarithmic intertwining operator of type* $\binom{W_3}{W_1 W_2}$ is a linear map

$$\mathcal{Y}(\cdot, x) \cdot : W_1 \otimes W_2 \rightarrow W_3[\log x]\{x\}, \quad (3.23)$$

or equivalently,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}(w_{(1)}, x) w_{(2)} = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} w_{(1)}^{\mathcal{Y}}_{n;k} w_{(2)} x^{-n-1} (\log x)^k \in W_3[\log x]\{x\} \quad (3.24)$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that the following conditions are satisfied: the *lower truncation condition*: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$$w_{(1)}^{\mathcal{Y}}_{n+m;k} w_{(2)} = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large, independently of } k; \quad (3.25)$$

the *Jacobi identity*:

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_3(v, x_1) \mathcal{Y}(w_{(1)}, x_2) w_{(2)} \\
& \quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_{(1)}, x_2) Y_2(v, x_1) w_{(2)} \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y_1(v, x_0) w_{(1)}, x_2) w_{(2)}
\end{aligned} \tag{3.26}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that the first term on the left-hand side is meaningful because of (3.25)); the $L(-1)$ -*derivative property*: for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx} \mathcal{Y}(w_{(1)}, x); \tag{3.27}$$

and the $\mathfrak{sl}(2)$ -*bracket relations*: for any $w_{(1)} \in W_1$,

$$[L(j), \mathcal{Y}(w_{(1)}, x)] = \sum_{i=0}^{j+1} \binom{j+1}{i} x^i \mathcal{Y}(L(j-i)w_{(1)}, x) \tag{3.28}$$

for $j = -1, 0$ and 1 (note that if V is in fact a conformal vertex algebra, this follows automatically from the Jacobi identity (3.26) by setting $v = \omega$ and then taking $\text{Res}_{x_0} \text{Res}_{x_1} x_1^{j+1}$).

Remark 3.12 We will sometimes write the Jacobi identity (3.26) as

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(v, x_1) \mathcal{Y}(w_{(1)}, x_2) w_{(2)} \\
& \quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_{(1)}, x_2) Y(v, x_1) w_{(2)} \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y(v, x_0) w_{(1)}, x_2) w_{(2)}
\end{aligned} \tag{3.29}$$

(dropping the subscripts on the module actions) for brevity.

Remark 3.13 The ordinary intertwining operators (as in, for example, [HL5]) among triples of modules for a vertex operator algebra are exactly the logarithmic intertwining operators that do not involve the formal variable $\log x$, except for our present relaxation of the lower truncation condition. The lower truncation condition that we use here can be equivalently stated as: For any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$, there is no nonzero term involving x^{n-m} appearing in $\mathcal{Y}(w_{(1)}, x) w_{(2)}$ when $m \in \mathbb{N}$ is large enough. In [HL5], the lower truncation condition in the definition of the notion of intertwining operator states: For any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$(w_{(1)})_n w_{(2)} = 0 \text{ for } n \text{ whose real part is sufficiently large.}$$

This is slightly stronger than the lower truncation condition that we use here, even if no $\log x$ is involved, when the powers of x in $\mathcal{Y}(w_{(1)}, x) w_{(2)}$ belong to infinitely many different congruence classes modulo \mathbb{Z} .

Remark 3.14 Given a generalized module (W, Y_W) for a Möbius (or conformal) vertex algebra V , the vertex operator map Y_W itself is clearly a logarithmic intertwining operator of type $\binom{W}{VW}$; in fact, it does not involve $\log x$ and its powers of x are all integers. In particular, taking (W, Y_W) to be (V, Y) itself, we have that the vertex operator map Y is a logarithmic intertwining operator of type $\binom{V}{VV}$ not involving $\log x$ and having only integral powers of x .

The logarithmic intertwining operators of a fixed type $\binom{W_3}{W_1 W_2}$ form a vector space.

Definition 3.15 In the setting of Definition 3.11, suppose in addition that V and W_1, W_2 and W_3 are strongly graded (recall Definitions 2.23 and 2.25). A logarithmic intertwining operator \mathcal{Y} as in Definition 3.11 is a *grading-compatible logarithmic intertwining operator* if for $\beta, \gamma \in \tilde{A}$ (recall Definition 2.25) and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$, $n \in \mathbb{C}$ and $k \in \mathbb{N}$, we have

$$w_{(1)} \mathcal{Y}_{n; k} w_{(2)} \in W_3^{(\beta+\gamma)}. \quad (3.30)$$

Remark 3.16 The term “grading-compatible” in Definition 3.15 refers to the \tilde{A} -gradings; any logarithmic intertwining operator is compatible with the \mathbb{C} -gradings of W_1, W_2 and W_3 , in view of Proposition 3.21(b) below.

Remark 3.17 Given a strongly graded generalized module (W, Y_W) for a strongly graded Möbius (or conformal) vertex algebra V , the vertex operator map Y_W is a grading-compatible logarithmic intertwining operator of type $\binom{W}{VW}$ not involving $\log x$ and having only integral powers of x . Taking (W, Y_W) in particular to be (V, Y) itself, we have that the vertex operator map Y is a grading-compatible logarithmic intertwining operator of type $\binom{V}{VV}$ not involving $\log x$ and having only integral powers of x .

In the strongly graded context (the main context for our tensor product theory), we will use the following notation and terminology, traditionally used in the setting of ordinary intertwining operators, as in [FHL]:

Definition 3.18 In the setting of Definition 3.15, the grading-compatible logarithmic intertwining operators of a fixed type $\binom{W_3}{W_1 W_2}$ form a vector space, which we denote by $\mathcal{V}_{W_1 W_2}^{W_3}$. We call the dimension of $\mathcal{V}_{W_1 W_2}^{W_3}$ the *fusion rule* for W_1, W_2 and W_3 and denote it by $N_{W_1 W_2}^{W_3}$.

Remark 3.19 In the strongly graded context, suppose that W_1, W_2 and W_3 in Definition 3.11 are expressed as finite direct sums of submodules. Then the space $\mathcal{V}_{W_1 W_2}^{W_3}$ can be naturally expressed as the corresponding (finite) direct sum of the spaces of (grading-compatible) logarithmic intertwining operators among the direct summands, and the fusion rule $N_{W_1 W_2}^{W_3}$ is thus the sum of the fusion rules for the direct summands.

Remark 3.20 As we shall point out in Remark 3.24 below, it turns out that the notion of fusion rule in Definition 3.18 agrees with the traditional notion, in the case of a vertex operator algebra and ordinary modules. The justification of this assertion uses Parts (b) and (c), or alternatively, Part (a), of the next proposition. Part (a), whose proof uses Lemma 3.9, shows how logarithmic intertwining operators yield expansions involving only finitely many powers of $\log x$. Part (b) is a generalization of formula (2.49).

Proposition 3.21 *Let W_1, W_2, W_3 be generalized modules for a Möbius (or conformal) vertex algebra V , and let $\mathcal{Y}(\cdot, x)$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1 W_2}$. Let $w_{(1)}$ and $w_{(2)}$ be homogeneous elements of W_1 and W_2 of generalized weights n_1 and $n_2 \in \mathbb{C}$, respectively, and let k_1 and k_2 be positive integers such that $(L(0) - n_1)^{k_1} w_{(1)} = 0$ and $(L(0) - n_2)^{k_2} w_{(2)} = 0$. Then we have:*

(a) ([Mi]) *For any $w'_{(3)} \in W_3^*$, $n_3 \in \mathbb{C}$ and $k_3 \in \mathbb{Z}_+$ such that $(L'(0) - n_3)^{k_3} w'_{(3)} = 0$,*

$$\begin{aligned} & \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle \\ & \in \mathbb{C}x^{n_3 - n_1 - n_2} \oplus \mathbb{C}x^{n_3 - n_1 - n_2} \log x \oplus \cdots \oplus \mathbb{C}x^{n_3 - n_1 - n_2} (\log x)^{k_1 + k_2 + k_3 - 3}. \end{aligned} \quad (3.31)$$

(b) *For any $n \in \mathbb{C}$ and $k \in \mathbb{N}$, $w_{(1)} \mathcal{Y}_{n; k} w_{(2)} \in W_3$ is homogeneous of generalized weight $n_1 + n_2 - n - 1$.*

(c) *Fix $n \in \mathbb{C}$ and $k \in \mathbb{N}$. For each $i, j \in \mathbb{N}$, let m_{ij} be a nonnegative integer such that*

$$(L(0) - n_1 - n_2 + n + 1)^{m_{ij}} ((L(0) - n_1)^i w_{(1)}) \mathcal{Y}_{n; k} (L(0) - n_2)^j w_{(2)} = 0.$$

Then for all $t \geq \max\{m_{ij} \mid 0 \leq i < k_1, 0 \leq j < k_2\} + k_1 + k_2 - 2$,

$$w_{(1)} \mathcal{Y}_{n; k+t} w_{(2)} = 0.$$

We will need the following lemma in the proof:

Lemma 3.22 *Let W_1, W_2, W_3 be generalized modules for a Möbius (or conformal) vertex algebra V . Let*

$$\begin{aligned} \mathcal{Y}(\cdot, x) : W_1 \otimes W_2 & \rightarrow W_3\{x, \log x\} \\ w_{(1)} \otimes w_{(2)} & \mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n, k \in \mathbb{C}} w_{(1)} \mathcal{Y}_{n; k} w_{(2)} x^{-n-1} (\log x)^k \end{aligned} \quad (3.32)$$

be a linear map that satisfies the $L(-1)$ -derivative property (3.27) and the $L(0)$ -bracket relation, that is, (3.28) with $j = 0$. Then for any $a, b, c \in \mathbb{C}$, $t \in \mathbb{N}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\begin{aligned} (L(0) - c)^t \mathcal{Y}(w_{(1)}, x)w_{(2)} &= \sum_{i, j, l \in \mathbb{N}, i+j+l=t} \frac{t!}{i!j!l!} \\ & \cdot \left(x \frac{d}{dx} - c + a + b \right)^l \mathcal{Y}((L(0) - a)^i w_{(1)}, x) (L(0) - b)^j w_{(2)}. \end{aligned} \quad (3.33)$$

Also, for any $a, b, n, k \in \mathbb{C}$, $t \in \mathbb{N}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, we have

$$\begin{aligned} & (L(0) - a - b + n + 1)^t (w_{(1)} \mathcal{Y}_{n; k} w_{(2)}) \\ &= t! \sum_{i, j, l \geq 0, i+j+l=t} \binom{k+l}{l} \left(\frac{(L(0) - a)^i}{i!} w_{(1)} \right) \mathcal{Y}_{n; k+l} \left(\frac{(L(0) - b)^j}{j!} w_{(2)} \right); \end{aligned} \quad (3.34)$$

in generating function form, this gives

$$\begin{aligned} & e^{y(L(0)-a-b+n+1)}(w_{(1)}^{\mathcal{Y}}_{n;k}w_{(2)}) \\ &= \sum_{l \in \mathbb{N}} \binom{k+l}{l} (e^{y(L(0)-a)}w_{(1)})^{\mathcal{Y}}_{n;k+l} (e^{y(L(0)-b)}w_{(2)})^{\mathcal{Y}}_l. \end{aligned} \quad (3.35)$$

Proof From (3.27) and (3.28) with $j = 0$ we have

$$\begin{aligned} L(0)\mathcal{Y}(w_{(1)}, x)w_{(2)} &= \mathcal{Y}(w_{(1)}, x)L(0)w_{(2)} \\ &+ x \frac{d}{dx} \mathcal{Y}(w_{(1)}, x)w_{(2)} + \mathcal{Y}(L(0)w_{(1)}, x)w_{(2)}. \end{aligned} \quad (3.36)$$

Hence

$$\begin{aligned} (L(0) - c)\mathcal{Y}(w_{(1)}, x)w_{(2)} &= \mathcal{Y}(w_{(1)}, x)(L(0) - b)w_{(2)} \\ &+ \left(x \frac{d}{dx} - c + a + b\right) \mathcal{Y}(w_{(1)}, x)w_{(2)} + \mathcal{Y}((L(0) - a)w_{(1)}, x)w_{(2)} \end{aligned}$$

for any complex numbers a, b and c . In view of the fact that the actions of $L(0)$ and d/dx commute with each other, this implies (3.33) essentially because of the expansion formula for powers of a sum of commuting operators, that is, for any commuting operators T_1, \dots, T_s and $t \in \mathbb{N}$,

$$(T_1 + \dots + T_s)^t = \sum_{i_1, \dots, i_s \in \mathbb{N}, i_1 + \dots + i_s = t} \frac{t!}{i_1! \dots i_s!} T_1^{i_1} \dots T_s^{i_s}. \quad (3.37)$$

On the other hand, by taking coefficient of $x^{-n-1}(\log x)^k$ in (3.36) we get

$$\begin{aligned} L(0)w_{(1)}^{\mathcal{Y}}_{n;k}w_{(2)} &= w_{(1)}^{\mathcal{Y}}_{n;k}L(0)w_{(2)} + (-n-1)w_{(1)}^{\mathcal{Y}}_{n;k}w_{(2)} \\ &+ (k+1)w_{(1)}^{\mathcal{Y}}_{n;k+1}w_{(2)} + (L(0)w_{(1)})^{\mathcal{Y}}_{n;k}w_{(2)}. \end{aligned}$$

So for any $a, b, n, k \in \mathbb{C}$,

$$\begin{aligned} (L(0) - a - b + n + 1)(w_{(1)}^{\mathcal{Y}}_{n;k}w_{(2)}) &= ((L(0) - a)w_{(1)})^{\mathcal{Y}}_{n;k}w_{(2)} \\ &+ w_{(1)}^{\mathcal{Y}}_{n;k}(L(0) - b)w_{(2)} + (k+1)w_{(1)}^{\mathcal{Y}}_{n;k+1}w_{(2)}. \end{aligned} \quad (3.38)$$

For $p, q \in \mathbb{N}$ and $n, k \in \mathbb{C}$, let us write

$$T_{p,k,q} = ((L(0) - a)^p w_{(1)})^{\mathcal{Y}}_{n;k} ((L(0) - b)^q w_{(2)}). \quad (3.39)$$

Then from (3.38) we see that for any $p, q \in \mathbb{N}$ and $a, b, n, k \in \mathbb{C}$,

$$(L(0) - a - b + n + 1)T_{p,k,q} = T_{p+1,k,q} + (k+1)T_{p,k+1,q} + T_{p,k,q+1}. \quad (3.40)$$

Hence by (3.37) we have

$$(L(0) - a - b + n + 1)^t T_{p,k,q} = t! \sum_{i,j,l \geq 0, i+j+l=t} \frac{(k+1)(k+2) \dots (k+l)}{i!j!l!} T_{p+i,k+l,q+j}$$

for any $a, b, n, k \in \mathbb{C}$ and $p, q \in \mathbb{N}$. In particular, by setting $p = q = 0$ we get (3.34), and (3.35) follows easily from (3.34) by multiplying by $y^t/t!$ and then summing over $t \in \mathbb{N}$. \square

Proof of Proposition 3.21 (a): Under the assumptions of the proposition, let us show that

$$\left\langle w'_{(3)}, \left(x \frac{d}{dx} - n_3 + n_1 + n_2 \right)^{k_3+k_1+k_2-2} \mathcal{Y}(w_{(1)}, x)w_{(2)} \right\rangle = 0 \quad (3.41)$$

by induction on $k_1 + k_2$.

For $k_1 = k_2 = 1$, from (3.33) with $a = n_1$, $b = n_2$, $c = n_3$ and $t = k_3$ we have

$$\begin{aligned} 0 &= \langle (L'(0) - n_3)^{k_3} w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle \\ &= \langle w'_{(3)}, (L(0) - n_3)^{k_3} \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle \\ &= \left\langle w'_{(3)}, \left(x \frac{d}{dx} - n_3 + n_1 + n_2 \right)^{k_3} \mathcal{Y}(w_{(1)}, x)w_{(2)} \right\rangle, \end{aligned}$$

which is (3.41) in the case $k_1 = k_2 = 1$.

Suppose that (3.41) is true for all the cases with smaller $k_1 + k_2$. Then from (3.33) with $a = n_1$, $b = n_2$, $c = n_3$ and $t = k_3 + k_1 + k_2 - 2$ we have

$$\begin{aligned} 0 &= \langle (L'(0) - n_3)^{k_3+k_1+k_2-2} w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle \\ &= \langle w'_{(3)}, (L(0) - n_3)^{k_3+k_1+k_2-2} \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle \\ &= \left\langle w'_{(3)}, \sum_{i,j,k \in \mathbb{N}, i+j+k=k_3+k_1+k_2-2} \frac{(k_3+k_1+k_2-2)!}{i!j!k!} \cdot \left(x \frac{d}{dx} - n_3 + n_1 + n_2 \right)^k \mathcal{Y}((L(0) - n_1)^i w_{(1)}, x) (L(0) - n_2)^j w_{(2)} \right\rangle \\ &= \left\langle w'_{(3)}, \left(x \frac{d}{dx} - n_3 + n_1 + n_2 \right)^{k_3+k_1+k_2-2} \mathcal{Y}(w_{(1)}, x)w_{(2)} \right\rangle, \end{aligned}$$

where the last equality uses the induction assumption for the pair of elements $(L(0) - n_1)^i w_{(1)}$ and $(L(0) - n_2)^j w_{(2)}$ for all $(i, j) \neq (0, 0)$. So (3.41) is established, that is, we have the formal differential equation

$$\left(x \frac{d}{dx} - n_3 + n_1 + n_2 \right)^{k_3+k_1+k_2-2} \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle = 0.$$

This implies (a) by Lemma 3.9.

(b): This follows from (3.34) with $a = n_1$, $b = n_2$ and the fact that for any $\bar{w}_{(1)} \in W_1$, $\bar{w}_{(2)} \in W_2$ and $\bar{n} \in \mathbb{C}$, there exists $K \in \mathbb{N}$ so that $(\bar{w}_{(1)})_{\bar{n}; \bar{k}}^{\mathcal{Y}} \bar{w}_{(2)} = 0$ for all $\bar{k} > K$, due to (3.23).

(c): Let us prove (c) by induction on $k_1 + k_2$ again. For $k_1 = k_2 = 1$, (3.34) with $a = n_1$ and $b = n_2$ gives

$$(L(0) - n_1 - n_2 + n + 1)^t (w_{(1)n;k}^{\mathcal{Y}} w_{(2)}) = ((k+t)!/k!) w_{(1)n;k+t}^{\mathcal{Y}} w_{(2)},$$

that is,

$$w_{(1)n;k+t}^{\mathcal{Y}} w_{(2)} = (k!/(k+t)!) (L(0) - n_1 - n_2 + n + 1)^t (w_{(1)n;k}^{\mathcal{Y}} w_{(2)}).$$

So for $t \geq m_{00}$, $w_{(1)n;k+t}^{\mathcal{Y}} w_{(2)} = 0$, proving the statement in case $k_1 = k_2 = 1$.

Suppose that the statement (c) is true for all smaller $k_1 + k_2$. Then for $(i, j) \neq (0, 0)$,

$$\left(\frac{(L(0) - n_1)^i}{i!} w_{(1)} \right)_{n;k+l}^{\mathcal{Y}} \left(\frac{(L(0) - n_2)^j}{j!} w_{(2)} \right) = 0$$

when $l \geq \max\{m_{i'j'} \mid i \leq i' < k_1, j \leq j' < k_2\} + (k_1 - i) + (k_2 - j) - 2$, and in particular, when $l \geq \max\{m_{i'j'} \mid 0 \leq i' < k_1, 0 \leq j' < k_2\} + k_1 + k_2 - i - j - 2$. But then for all $t \geq \max\{m_{ij} \mid 0 \leq i < k_1, 0 \leq j < k_2\} + k_1 + k_2 - 2$, (3.34) gives $0 = ((k+t)!/k!) w_{(1)n;k+t}^{\mathcal{Y}} w_{(2)}$, proving what we need. \square

The following corollary is immediate from Proposition 3.21(b):

Corollary 3.23 *Let V be a Möbius (or conformal) vertex algebra and let W_1, W_2 and W_3 be generalized V -modules whose weights are all congruent modulo \mathbb{Z} to complex numbers h_1, h_2 and h_3 , respectively. (For example, W_1, W_2 and W_3 might be indecomposable; recall Remark 2.20.) Let $\mathcal{Y}(\cdot, x) \cdot$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then all powers of x in $\mathcal{Y}(\cdot, x) \cdot$ are congruent modulo \mathbb{Z} to $h_3 - h_1 - h_2$. \square*

Remark 3.24 Let W_1, W_2 and W_3 be (ordinary) modules for a Möbius (or conformal) vertex algebra V . Then any logarithmic intertwining operator of type $\binom{W_3}{W_1 W_2}$ is just an ordinary intertwining operator of this type, i.e., it does not involve $\log x$. This clearly follows from Proposition 3.21(b) and (c), where k_1 and k_2 are chosen to be 1, k is chosen to be 0, and m_{00} is chosen to be 1. It also follows, alternatively, from Proposition 3.21(a). As a result, for V a vertex operator algebra (viewed as a conformal vertex algebra strongly graded with respect to the trivial group; recall Remark 2.24) and W_1, W_2 and W_3 V -modules in the sense of Remark 2.27, the notion of fusion rule defined in this work (recall Definition 3.18) coincides with the notion of fusion rule defined in, for example, [HL5] (except for the minor issue of the truncation condition for an intertwining operator, discussed in Remark 3.13).

Remark 3.25 Our definition of logarithmic intertwining operator is identical to that in [Mi] (in case V is a vertex operator algebra) except that in [Mi], a logarithmic intertwining operator \mathcal{Y} of type $\binom{W_3}{W_1 W_2}$ is required to be a linear map $W_1 \rightarrow \text{Hom}(W_2, W_3)\{x\}[\log x]$, instead of as in (3.23), and the lower truncation condition (3.25) is replaced by: For any $w_{(1)} \in W_1, w_{(2)} \in W_2$ and $k \in \mathbb{N}$,

$$w_{(1)n;k}^{\mathcal{Y}} w_{(2)} = 0 \text{ for } n \text{ whose real part is sufficiently large.}$$

Given generalized V -modules W_1 , W_2 and W_3 , suppose that for each $i = 1, 2, 3$, there exists some $K_i \in \mathbb{Z}_+$ such that $(L(0) - L(0)_s)^{K_i} W_i = 0$ (this is satisfied by many interesting examples and is assumed in [Mi] for all generalized modules under consideration). Then for any logarithmic intertwining operator \mathcal{Y} , any homogeneous elements $w_{(1)} \in W_{1[n_1]}$, $w_{(2)} \in W_{2[n_2]}$, $n_1, n_2 \in \mathbb{C}$, any $n \in \mathbb{C}$ and any $k \in \mathbb{N}$, all the m_{ij} 's in Proposition 3.21(c) can be chosen to be no greater than K_3 , while k_1 and k_2 can be chosen to be no greater than K_1 and K_2 , respectively. Proposition 3.21(c) thus implies that the largest power of $\log x$ that is involved in \mathcal{Y} is no greater than $K_1 + K_2 + K_3 - 3$. In particular, \mathcal{Y} maps W_1 to $\text{Hom}(W_2, W_3)\{x\}[\log x]$, and in fact, we even have that $K_1 + K_2 + K_3 - 3$ is a global bound on the powers of $\log x$, independently of $w_{(1)} \in W_1$, so that

$$\mathcal{Y}(\cdot, x) \in \text{Hom}(W_1 \otimes W_2, W_3)\{x\}[\log x]. \quad (3.42)$$

Remark 3.26 Given a logarithmic intertwining operator \mathcal{Y} as in (3.24), set

$$\mathcal{Y}^{(k)}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} w_{(1)}^{\mathcal{Y}}_{n; k} w_{(2)} x^{-n-1}$$

for $k \in \mathbb{N}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, so that

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{k \in \mathbb{N}} \mathcal{Y}^{(k)}(w_{(1)}, x)w_{(2)} (\log x)^k.$$

By taking the coefficients of the powers of $\log x_2$ and $\log x$ in (3.26) and (3.28), respectively, we see that each $\mathcal{Y}^{(k)}$ satisfies the Jacobi identity and the $\mathfrak{sl}(2)$ -bracket relations. On the other hand, taking the coefficients of the powers of $\log x$ in (3.27) gives

$$\mathcal{Y}^{(k)}(L(-1)w_{(1)}, x) = \frac{d}{dx} \mathcal{Y}^{(k)}(w_{(1)}, x) + \frac{k+1}{x} \mathcal{Y}^{(k+1)}(w_{(1)}, x) \quad (3.43)$$

for any $k \in \mathbb{N}$ and $w_{(1)} \in W_1$. So $\mathcal{Y}^{(k)}$ does not in general satisfy the $L(-1)$ -derivative property. (If $\mathcal{Y}^{(k+1)} = 0$, then $\mathcal{Y}^{(k)}$ of course does satisfy the $L(-1)$ -derivative property and so is an (ordinary) intertwining operator; this certainly happens for $k = 0$ in the context of Remark 3.24 and for $k = K_1 + K_2 + K_3 - 3$ in the context of Remark 3.25.) However, in the following we will see that suitable formal linear combinations of certain modifications of $\mathcal{Y}^{(k)}$ (depending on $t \in \mathbb{N}$; see below) form a sequence of logarithmic intertwining operators.

Remark 3.27 Given a logarithmic intertwining operator \mathcal{Y} , let us write

$$\begin{aligned} \mathcal{Y}(w_{(1)}, x)w_{(2)} &= \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} w_{(1)}^{\mathcal{Y}}_{n; k} w_{(2)} x^{-n-1} (\log x)^k \\ &= \sum_{\mu \in \mathbb{C}/\mathbb{Z}} \sum_{\bar{n}=\mu} \sum_{k \in \mathbb{N}} w_{(1)}^{\mathcal{Y}}_{n; k} w_{(2)} x^{-n-1} (\log x)^k \end{aligned}$$

for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, where \bar{n} denotes the equivalence class of n in \mathbb{C}/\mathbb{Z} . By extracting summands corresponding to the same congruence class modulo \mathbb{Z} of the powers of x in (3.26), (3.27) and (3.28) we see that for each $\mu \in \mathbb{C}/\mathbb{Z}$,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}^\mu(w_{(1)}, x)w_{(2)} = \sum_{\bar{n}=\mu} \sum_{k \in \mathbb{N}} w_{(1)}^{\mathcal{Y}}_{n;k} w_{(2)} x^{-n-1} (\log x)^k \quad (3.44)$$

still defines a logarithmic intertwining operator. In the strongly graded case, if \mathcal{Y} is grading-compatible, then so is the operator \mathcal{Y}^μ in (3.44). Conversely, suppose that we are given a family of logarithmic intertwining operators $\{\mathcal{Y}^\mu | \mu \in \mathbb{C}/\mathbb{Z}\}$ parametrized by $\mu \in \mathbb{C}/\mathbb{Z}$ such that the powers of x in \mathcal{Y}^μ are restricted as in (3.44). Then the formal sum $\sum_{\mu \in \mathbb{C}/\mathbb{Z}} \mathcal{Y}^\mu$ is well defined and is a logarithmic intertwining operator. In the strongly graded case, if each \mathcal{Y}^μ is grading-compatible, then so is this sum.

In the setting of Definition 3.11, for any integer p , set

$$\mathcal{Y}(w_{(1)}, e^{2\pi ip}x)w_{(2)} = \mathcal{Y}(w_{(1)}, y)w_{(2)} \Big|_{y^n = e^{2\pi ipn}x^n, (\log y)^k = (2\pi ip + \log x)^k, n \in \mathbb{C}, k \in \mathbb{N}}. \quad (3.45)$$

This is in fact a well-defined element of $W_3[\log x]\{x\}$. Note that this element certainly depends on p , not just on $e^{2\pi ip}$ ($= 1$). This substitution, which can be thought of as “ $x \mapsto e^{2\pi ip}x$,” will be considered in a more general form in (3.74) below.

Remark 3.28 It is clear that in Definition 3.11, for any integer p , all the axioms are formally invariant under the substitution $x \mapsto e^{2\pi ip}x$ given by (3.45). That is, if we apply this substitution to each axiom, the axiom keeps the same form, with the operator $\mathcal{Y}(\cdot, x)$ replaced by $\mathcal{Y}(\cdot, e^{2\pi ip}x)$. For example, for the Jacobi identity (3.26), we perform the substitution $x_2 \mapsto e^{2\pi ip}x_2$; the formal delta-functions remain unchanged because they involve only integral powers of x_2 and no logarithms. It follows that $\mathcal{Y}(\cdot, e^{2\pi ip}x)$ is again a logarithmic intertwining operator.

From Remark 3.28, for any $\mu \in \mathbb{C}/\mathbb{Z}$ and logarithmic intertwining operator \mathcal{Y}^μ as in (3.44), the linear map defined by

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}^\mu(w_{(1)}, e^{2\pi ip}x)w_{(2)} = \sum_{\bar{n}=\mu} \sum_{k \in \mathbb{N}} w_{(1)}^{\mathcal{Y}}_{n;k} w_{(2)} e^{2\pi ip(-n-1)} x^{-n-1} (\log x + 2\pi ip)^k$$

is also a logarithmic intertwining operator. In the strongly graded case, if the operator in (3.44) is grading-compatible, then so is this one. The right-hand side above can be written as

$$\begin{aligned} & e^{-2\pi ip\mu} \sum_{\bar{n}=\mu} \sum_{k \in \mathbb{N}} w_{(1)}^{\mathcal{Y}}_{n;k} w_{(2)} x^{-n-1} \sum_{t \in \mathbb{N}} \binom{k}{t} (\log x)^{k-t} (2\pi ip)^t \\ &= e^{-2\pi ip\mu} \sum_{t \in \mathbb{N}} (2\pi ip)^t \sum_{k \in \mathbb{N}} \binom{k+t}{t} \sum_{\bar{n}=\mu} w_{(1)}^{\mathcal{Y}}_{n;k+t} w_{(2)} x^{-n-1} (\log x)^k \end{aligned}$$

(the coefficient of each power of x being a finite sum over t and k). We now have:

Proposition 3.29 *Let W_1, W_2, W_3 be generalized modules for a Möbius (or conformal) vertex algebra V , and let $\mathcal{Y}(\cdot, x)\cdot$ be a logarithmic intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right)$. For $\mu \in \mathbb{C}/\mathbb{Z}$ and $t \in \mathbb{N}$, define $\mathcal{X}_t^\mu : W_1 \otimes W_2 \rightarrow W_3[\log x]\{x\}$ by:*

$$\mathcal{X}_t^\mu : w_{(1)} \otimes w_{(2)} \mapsto \sum_{k \in \mathbb{N}} \binom{k+t}{t} \sum_{\bar{n}=\mu} w_{(1)}^{\mathcal{Y}}_{n; k+t} w_{(2)} x^{-n-1} (\log x)^k.$$

Then each \mathcal{X}_t^μ is a logarithmic intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right)$. In particular, the operator \mathcal{X}_t defined by

$$\mathcal{X}_t : w_{(1)} \otimes w_{(2)} \mapsto \sum_{k \in \mathbb{N}} \binom{k+t}{t} \sum_{n \in \mathbb{C}} w_{(1)}^{\mathcal{Y}}_{n; k+t} w_{(2)} x^{-n-1} (\log x)^k \quad (3.46)$$

is a logarithmic intertwining operator of the same type. In the strongly graded case, if \mathcal{Y} is grading-compatible, then so are \mathcal{X}_t^μ and \mathcal{X}_t .

Proof From the above we see that for any integer p , $e^{-2\pi i p \mu} \sum_{t \in \mathbb{N}} (2\pi i p)^t \mathcal{X}_t^\mu$, and hence

$$\sum_{t \in \mathbb{N}} (2\pi i p)^t \mathcal{X}_t^\mu, \quad (3.47)$$

is a logarithmic intertwining operator. Let us now prove that \mathcal{X}_m^μ is a logarithmic intertwining operator by induction on m . The $m = 0$ case follows immediately from setting $p = 0$ in (3.47). Suppose that $\mathcal{X}_0^\mu, \dots, \mathcal{X}_{m-1}^\mu$ are all logarithmic intertwining operators. Then for any integer p ,

$$\sum_{t \geq m} (2\pi i p)^t \mathcal{X}_t^\mu = \sum_{t \in \mathbb{N}} (2\pi i p)^t \mathcal{X}_t^\mu - \sum_{t=0}^{m-1} (2\pi i p)^t \mathcal{X}_t^\mu$$

is also a logarithmic intertwining operator. Dividing this by $(2\pi i p)^m$ and then setting $p = 0$ we see that \mathcal{X}_m^μ is also a logarithmic intertwining operator. The second assertion follows from Remark 3.27, and the last assertion is clear. \square

Remark 3.30 Let $W_i, W^i, i = 1, 2, 3$, be generalized modules for a Möbius (or conformal) vertex algebra V . If $\mathcal{Y}(\cdot, x)\cdot$ is a logarithmic intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right)$ and $\sigma_1 : W^1 \rightarrow W_1, \sigma_2 : W^2 \rightarrow W_2$ and $\sigma_3 : W_3 \rightarrow W^3$ are V -module homomorphisms, then it is easy to see that $\sigma_3 \mathcal{Y}(\sigma_1 \cdot, x) \sigma_2 \cdot$ is a logarithmic intertwining operator of type $\left(\begin{smallmatrix} W^3 \\ W^1 W^2 \end{smallmatrix}\right)$. In the strongly graded case, if \mathcal{Y} is grading-compatible, then so is $\sigma_3 \mathcal{Y}(\sigma_1 \cdot, x) \sigma_2 \cdot$ (recall from Remark 2.28 that each σ_j preserves the \tilde{A} -grading). That is, in categorical language, with \mathcal{C} a full subcategory of the category of either \mathcal{M} (the category of V -modules; recall Notation 2.36), \mathcal{GM} (the category of generalized V -modules), \mathcal{M}_{sg} (the category of strongly graded V -modules) or \mathcal{GM}_{sg} (the category of strongly graded generalized V -modules), the correspondence from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to the category **Vect** of vector spaces given by $(W_1, W_2, W_3) \mapsto \mathcal{V}_{W_1 W_2}^{W_3}$ is functorial in the third slot and cofunctorial in the first two slots.

Remark 3.31 Now recall from Remark 2.21 that $L(0) - L(0)_s$ commutes with the actions of both V and $\mathfrak{sl}(2)$. So $(L(0) - L(0)_s)^i$ is a V -module homomorphism from a generalized module to itself for any $i \in \mathbb{N}$, and this remains true in the strongly graded case. Hence by Remark 3.30, given any logarithmic intertwining operator $\mathcal{Y}(\cdot, x) \cdot$ as in (3.23) and any $i, j, k \in \mathbb{N}$,

$$(L(0) - L(0)_s)^k \mathcal{Y}((L(0) - L(0)_s)^i \cdot, x) (L(0) - L(0)_s)^j \cdot \quad (3.48)$$

is again a logarithmic intertwining operator, and in the strongly graded case, if \mathcal{Y} is grading-compatible, so is this operator. In the next remark we will see that the logarithmic intertwining operators (3.46) are just linear combinations of these.

Remark 3.32 Let W_1, W_2, W_3 and \mathcal{Y} be as above and let $w_{(1)} \in W_{1[n_1]}$ and $w_{(2)} \in W_{2[n_2]}$ for some complex numbers n_1 and n_2 . Fixing $n \in \mathbb{C}$ and using the notation $T_{p,k,q}$ in (3.39) with $a = n_1$ and $b = n_2$, we rewrite formula (3.40) in the proof of Lemma 3.22 as

$$(k+1)T_{p,k+1,q} = (L(0) - n_1 - n_2 + n + 1)T_{p,k,q} - T_{p+1,k,q} - T_{p,k,q+1}$$

for any $p, q, k \in \mathbb{N}$. From this, by (3.37) we see that for any $t \in \mathbb{N}$,

$$\binom{k+t}{t} T_{p,k+t,q} = \sum_{i,j,l \in \mathbb{N}, i+j+l=t} \frac{1}{i!j!l!} (-1)^{i+j} (L(0) - n_1 - n_2 + n + 1)^l T_{p+i,k,q+j}.$$

Setting $p = q = 0$ we get

$$\begin{aligned} \binom{k+t}{t} w_{(1)n; k+t}^{\mathcal{Y}} w_{(2)} &= \sum_{i,j,l \in \mathbb{N}, i+j+l=t} \frac{1}{i!j!l!} (-1)^{i+j} \\ &\cdot (L(0) - n_1 - n_2 + n + 1)^l ((L(0) - n_1)^i w_{(1)})_{n;k}^{\mathcal{Y}} (L(0) - n_2)^j w_{(2)} \end{aligned} \quad (3.49)$$

for any $t \in \mathbb{N}$. (Note that this formula gives an alternate proof of Proposition 3.21(c).) Now multiplying by $x^{-n-1}(\log x)^k$ and then summing over $n \in \mathbb{C}$ and over $k \in \mathbb{N}$ we see that for every $t \in \mathbb{N}$ the intertwining operator \mathcal{X}_t in (3.46) is a linear combination of intertwining operators of the form (3.48).

In preparation for generalizing basic results from [FHL] on intertwining operators to the logarithmic case, we need to generalize more of the basic tools. We now define operators “ $x^{\pm L(0)}$ ” for generalized modules, in the natural way:

Definition 3.33 Let W be a generalized module for a Möbius (or conformal) vertex algebra. We define $x^{\pm L(0)} : W \rightarrow W\{x\}[\log x] \subset W[\log x]\{x\}$ as follows: For any $w \in W_{[n]}$ ($n \in \mathbb{C}$), define

$$x^{\pm L(0)} w = x^{\pm n} e^{\pm \log x (L(0) - n)} w \quad (3.50)$$

(note that the local nilpotence of $L(0) - n$ on $W_{[n]}$ insures that the formal exponential series terminates) and then extend linearly to all $w \in W$. (Of course, we could also write

$$x^{\pm L(0)} = x^{\pm L(0)_s} e^{\pm \log x (L(0) - L(0)_s)}, \quad (3.51)$$

using the notation $L(0)_s$.) We also define operators $x^{\pm L'(0)}$ on W^* by the condition that for all $w' \in W^*$ and $w \in W$,

$$\langle x^{\pm L'(0)} w', w \rangle = \langle w', x^{\pm L(0)} w \rangle \in \mathbb{C}\{x\}[\log x], \quad (3.52)$$

so that $x^{\pm L'(0)} : W^* \rightarrow W^*\{x\}[[\log x]]$.

Remark 3.34 Note that these definitions are (of course) compatible with the usual definitions if W is just an (ordinary) module. In formula (3.50), $x^{\pm L(0)}$ is defined in a naturally factored form reminiscent of the factorization invoked in Remark 3.10 (providing counterexamples there); the symbol $x^{\pm(L(0)-n)}$ is given meaning by its replacement by $e^{\pm \log x(L(0)-n)}$. Also note that in case both V and W are strongly graded, the definition of $x^{\pm L'(0)}$ given by (3.52), when applied to the subspace W' of W^* , coincides with the definition of $x^{\pm L'(0)}$ given by (3.50) induced from the contragredient module action of V on W' . (Recall Theorem 2.34.)

Remark 3.35 Note that for $w \in W_{[n]}$, by definition we have

$$x^{\pm L(0)} w = x^{\pm n} \sum_{i \in \mathbb{N}} \frac{(L(0) - n)^i w}{i!} (\pm \log x)^i \in x^{\pm n} W_{[n]}[\log x]. \quad (3.53)$$

It is also handy to have that for any $w \in W$,

$$x^{L(0)} x^{-L(0)} w = w = x^{-L(0)} x^{L(0)} w, \quad (3.54)$$

which is clear from definition. Later we will also need the formula

$$\frac{d}{dx} x^{\pm L(0)} w = \pm x^{-1} x^{\pm L(0)} L(0) w \quad (3.55)$$

for any $w \in W$, i.e.,

$$x \frac{d}{dx} x^{\pm L(0)} w = \pm x^{\pm L(0)} L(0) w, \quad (3.56)$$

or equivalently,

$$\left(x \frac{d}{dx} \mp L(0) \right) x^{\pm L(0)} w = 0 \quad (3.57)$$

(cf. Lemma 3.9 and Remark 3.10). This can be proved by directly checking that for w homogeneous of generalized weight n ,

$$\frac{d}{dx} e^{\pm \log x(L(0)-n)} w = \pm x^{-1} e^{\pm \log x(L(0)-n)} (L(0) - n) w,$$

and hence for such w ,

$$\begin{aligned} \frac{d}{dx} x^{\pm L(0)} w &= \frac{d}{dx} (x^{\pm n} e^{\pm \log x(L(0)-n)} w) \\ &= \pm n x^{\pm n-1} e^{\pm \log x(L(0)-n)} w \pm x^{\pm n-1} e^{\pm \log x(L(0)-n)} (L(0) - n) w \\ &= \pm x^{\pm n-1} e^{\pm \log x(L(0)-n)} L(0) w = \pm x^{-1} x^{\pm L(0)} L(0) w. \end{aligned}$$

In the statement (and proof) of the next result, we shall use expressions of the type

$$\begin{aligned}(1-x)^{L(0)} &= \sum_{k \in \mathbb{N}} \binom{L(0)}{k} (-x)^k \\ &= \sum_{k \in \mathbb{N}} \frac{L(0)(L(0)-1) \cdots (L(0)-k+1)}{k!} (-x)^k,\end{aligned}$$

which also equals

$$e^{L(0) \log(1-x)},$$

as well as expressions involving

$$(x(1-yx)^{-1})^n = \sum_{k \in \mathbb{N}} \binom{-n}{k} x^n (-yx)^k$$

for $n \in \mathbb{C}$ and

$$\begin{aligned}\log(x(1-yx)^{-1}) &= \log x + \log(1-yx)^{-1} \\ &= \log x + \sum_{k \geq 1} \frac{1}{k} (yx)^k.\end{aligned}$$

We can now state and prove generalizations to logarithmic intertwining operators of three standard formulas for (ordinary) intertwining operators, namely, formulas (5.4.21), (5.4.22) and (5.4.23) of [FHL]. The result is (see also [Mi] for Parts (a) and (b)):

Proposition 3.36 *Let \mathcal{Y} be a logarithmic intertwining operator of type $\binom{W_3}{W_1 W_2}$ and let $w \in W_1$. Then*

$$(a) \quad e^{yL(-1)} \mathcal{Y}(w, x) e^{-yL(-1)} = \mathcal{Y}(e^{yL(-1)} w, x) = \mathcal{Y}(w, x+y) \quad (3.58)$$

(recall (3.5))

$$(b) \quad y^{L(0)} \mathcal{Y}(w, x) y^{-L(0)} = \mathcal{Y}(y^{L(0)} w, xy) \quad (3.59)$$

(recall (3.8))

$$(c) \quad e^{yL(1)} \mathcal{Y}(w, x) e^{-yL(1)} = \mathcal{Y}(e^{y(1-yx)L(1)} (1-yx)^{-2L(0)} w, x(1-yx)^{-1}). \quad (3.60)$$

Proof From (3.28) with $j = -1$ we see that for any $w \in W_1$,

$$L(-1) \mathcal{Y}(w, x) = \mathcal{Y}(L(-1)w, x) + \mathcal{Y}(w, x) L(-1).$$

This implies

$$\frac{y^n(L(-1))^n}{n!}\mathcal{Y}(w, x) = \sum_{i, j \in \mathbb{N}, i+j=n} \mathcal{Y}\left(\frac{y^i(L(-1))^i}{i!}w, x\right) \frac{y^j(L(-1))^j}{j!}$$

for any $n \in \mathbb{N}$, where y is a new formal variable. Summing over $n \in \mathbb{N}$ we see that for any $w \in W_1$,

$$e^{yL(-1)}\mathcal{Y}(w, x) = \mathcal{Y}(e^{yL(-1)}w, x)e^{yL(-1)},$$

and hence

$$e^{yL(-1)}\mathcal{Y}(w, x)e^{-yL(-1)} = \mathcal{Y}(e^{yL(-1)}w, x) = e^{y\frac{d}{dx}}\mathcal{Y}(w, x) = \mathcal{Y}(w, x+y), \quad (3.61)$$

where in the last equality we have used (3.10). Note that all the expressions in (3.61) remain well defined if we replace y by any element of $y\mathbb{C}[x][[y]]$. Thus by Remark 3.7, (3.61) still holds if we replace y by any such element.

For (b), note that for homogeneous $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, by (3.35) with the formal variable y replaced by the formal variable $\log y$, we get (recalling Proposition 3.21(b) and (3.50))

$$y^{L(0)}(w_{(1)}\mathcal{Y}_{n;k}w_{(2)}) = \sum_{l \in \mathbb{N}} \binom{k+l}{k} (y^{L(0)}w_{(1)})\mathcal{Y}_{n;k+l}(y^{L(0)}w_{(2)})y^{-n-1}(\log y)^l$$

for any $n \in \mathbb{C}$ and $k \in \mathbb{N}$. Multiplying this by $x^{-n-1}(\log x)^k$, summing over $n \in \mathbb{C}$ and $k \in \mathbb{N}$ and using (3.8) we get

$$y^{L(0)}\mathcal{Y}(w_{(1)}, x)w_{(2)} = \mathcal{Y}(y^{L(0)}w_{(1)}, xy)y^{L(0)}w_{(2)}.$$

Formula (3.59) then follows from (3.54).

Finally we prove (c). From (3.28) with $j = 1$ we see that for any $w \in W_1$,

$$L(1)\mathcal{Y}(w, x) = \mathcal{Y}((L(1) + 2xL(0) + x^2L(-1))w, x) + \mathcal{Y}(w, x)L(1).$$

This implies that

$$e^{yL(1)}\mathcal{Y}(w, x) = \mathcal{Y}(e^{y(L(1)+2xL(0)+x^2L(-1))}w, x)e^{yL(1)},$$

or

$$e^{yL(1)}\mathcal{Y}(w, x)e^{-yL(1)} = \mathcal{Y}(e^{y(L(1)+2xL(0)+x^2L(-1))}w, x).$$

Using the identity

$$e^{y(L(1)+2xL(0)+x^2L(-1))} = e^{yx^2(1-yx)^{-1}L(-1)}e^{y(1-yx)L(1)}(1-yx)^{-2L(0)},$$

whose proof is exactly the same as the proof of formula (5.2.41) of [FHL], we obtain

$$e^{yL(1)}\mathcal{Y}(w, x)e^{-yL(1)} = \mathcal{Y}(e^{yx^2(1-yx)^{-1}L(-1)}e^{y(1-yx)L(1)}(1-yx)^{-2L(0)}w, x). \quad (3.62)$$

But by (3.61) with y replaced by $yx^2(1-yx)^{-1}$, the right-hand side of (3.62) is equal to

$$\mathcal{Y}(e^{y(1-yx)L(1)}(1-yx)^{-2L(0)}w, x+yx^2(1-yx)^{-1}). \quad (3.63)$$

Since

$$(x+yx^2(1-yx)^{-1})^n = (x(1-yx)^{-1})^n$$

for $n \in \mathbb{C}$ and

$$\log(x+yx^2(1-yx)^{-1}) = \log(x(1-yx)^{-1}),$$

(3.63) is equal to the right-hand side of (3.60), proving (3.60). \square

Remark 3.37 The following formula, also a generalization of the corresponding formula in the ordinary case (see (2.69)), will be needed: For $j = -1, 0, 1$,

$$x^{L(0)}L(j)x^{-L(0)} = x^{-j}L(j). \quad (3.64)$$

To prove this, we first observe that for any $m \in \mathbb{C}$, $[L(0) - m, L(j)] = -jL(j)$ implies that

$$e^{\log x(L(0)-m)}L(j)e^{-\log x(L(0)-m)} = e^{-j \log x}L(j).$$

Hence, for a generalized module element w homogeneous of generalized weight n ,

$$\begin{aligned} x^{L(0)}L(j)w &= x^{n-j}e^{\log x(L(0)-n+j)}L(j)w \\ &= x^{n-j}e^{-j \log x}L(j)e^{\log x(L(0)-n+j)}w \\ &= x^{n-j}L(j)e^{\log x(L(0)-n)}w \\ &= x^{-j}L(j)x^{L(0)}w, \end{aligned}$$

and (3.64) then follows immediately from (3.54).

Remark 3.38 From (3.64) we see that

$$x^{L(0)}e^{yL(j)}x^{-L(0)} = e^{yx^{-j}L(j)}. \quad (3.65)$$

For an ordinary module W for a vertex operator algebra and any $a \in \mathbb{C}$, the operator $e^{aL(0)}$ on W is defined by

$$e^{aL(0)}w = e^{ah}w \quad (3.66)$$

for any homogeneous $w \in W_{(h)}$, $h \in \mathbb{C}$ and then by linear extension to any $w \in W$. More generally, for a generalized module W for a Möbius (or conformal) vertex algebra and any $a \in \mathbb{C}$, we define the operator $e^{aL(0)}$ on W by

$$e^{aL(0)}w = e^{ah}e^{a(L(0)-h)}w \quad (3.67)$$

for any homogeneous $w \in W_{[h]}$, $h \in \mathbb{C}$ and then by linear extension to all $w \in W$. (Note that for a formal variable x , we already have $e^{xL(0)}w = e^{hx}e^{x(L(0)-h)}w$.) From the definition,

$$e^{aL(0)}e^{-aL(0)}w = w. \quad (3.68)$$

Recalling Remark 2.21 for the notation $L(0)_s$, we see that

$$e^{aL(0)} = e^{aL(0)_s}e^{a(L(0)-L(0)_s)} \text{ on } W, \quad (3.69)$$

where the exponential series $e^{a(L(0)-L(0)_s)}$ terminates on each element of W .

Remark 3.39 The operators defined in (3.66) and (3.67) can be alternatively defined or viewed as the (analytically) convergent sums of the indicated exponential series of operators; these operators act on the (finite-dimensional) subspaces of W generated by the repeated action of $L(0)$ on homogeneous vectors $w \in W$.

Remark 3.40 The operator $e^{a(L(0)-L(0)_s)}$ on W is a V -homomorphism, in view of Remark 2.21 (cf. Remark 3.31). Let r be an integer. Then $e^{2\pi irL(0)_s}$ is also a V -homomorphism, by Remark 2.20. Thus for $r \in \mathbb{Z}$, $e^{2\pi irL(0)}$ is a V -homomorphism. In the strongly graded case, all of these V -homomorphisms are grading-preserving.

Remark 3.41 We now recall some identities about the action of $\mathfrak{sl}(2)$ on any of its modules. For convenience we put them in the following form:

$$e^{xL(-1)} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} e^{-xL(-1)} = \begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ x^2 & -2x & 1 \end{pmatrix} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} \quad (3.70)$$

$$e^{xL(0)} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} e^{-xL(0)} = \begin{pmatrix} e^x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-x} \end{pmatrix} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} \quad (3.71)$$

$$e^{xL(1)} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} e^{-xL(1)} = \begin{pmatrix} 1 & 2x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix}. \quad (3.72)$$

Formula (3.71) follows from (5.2.12) and (5.2.13) in [FHL]; Formula (3.72) follows from (5.2.14) in [FHL] and $[L(1), L(-1)] = 2L(0)$; and formula (3.70) follows from (3.72) and the fact that

$$L(-1) \mapsto L(1), \quad L(0) \mapsto -L(0), \quad L(1) \mapsto L(-1)$$

is a Lie algebra automorphism of $\mathfrak{sl}(2)$.

Remark 3.42 It is convenient to note that the $\mathfrak{sl}(2)$ -bracket relations (3.28) are equivalent to

$$\mathcal{Y}(L(j)w_{(1)}, x) = \sum_{i=0}^{j+1} \binom{j+1}{i} (-x)^i [L(j-i), \mathcal{Y}(w_{(1)}, x)] \quad (3.73)$$

for $j = -1, 0$ and 1 . This can be easily checked by writing (3.28) as

$$\begin{pmatrix} [L(-1), \mathcal{Y}(w_{(1)}, x)] \\ [L(0), \mathcal{Y}(w_{(1)}, x)] \\ [L(1), \mathcal{Y}(w_{(1)}, x)] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & 2x & 1 \end{pmatrix} \begin{pmatrix} \mathcal{Y}(L(-1)w_{(1)}, x) \\ \mathcal{Y}(L(0)w_{(1)}, x) \\ \mathcal{Y}(L(1)w_{(1)}, x) \end{pmatrix}$$

and then multiplying this by the inverse of the invertible matrix on the right-hand side, obtained by replacing x by $-x$. (Of course, in the case where V is conformal, this equivalence is already encoded in the symmetry of the Jacobi identity.)

We have already defined the natural process of multiplying the formal variable in a logarithmic intertwining operator by $e^{2\pi ip}$ for $p \in \mathbb{Z}$ (recall (3.45)), and this process yields another logarithmic intertwining operator (recall Remark (3.28)). It is natural to generalize this substitution process to that of multiplying the formal variable in a logarithmic intertwining operator by the exponential e^ζ of any complex number ζ . As in the special case $\zeta = 2\pi ip$, the process will depend on ζ , not just on e^ζ , but we will still find it convenient to use the shorthand symbol e^ζ in our notation for the process. In Section 7 of [HL6], we introduced this procedure in the case of ordinary (nonlogarithmic) intertwining operators, and we now carry it out in the general logarithmic case. We are about to use this substitution mostly for $\zeta = (2r+1)\pi i$, $r \in \mathbb{Z}$.

Let (W_1, Y_1) , (W_2, Y_2) and (W_3, Y_3) be generalized modules for a Möbius (or conformal) vertex algebra V . Let \mathcal{Y} be a logarithmic intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right)$. For any complex number ζ and any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, set

$$\mathcal{Y}(w_{(1)}, e^\zeta x)w_{(2)} = \mathcal{Y}(w_{(1)}, y)w_{(2)} \Big|_{y^n = e^{\zeta n} x^n, (\log y)^k = (\zeta + \log x)^k, n \in \mathbb{C}, k \in \mathbb{N}}, \quad (3.74)$$

a well-defined element of $W_3[\log x]\{x\}$. Note that this element indeed depends on ζ , not just on e^ζ .

Remark 3.43 In Section 4 below we will take the further step of specializing the formal variable x to 1 (or equivalently, the formal variable y to e^ζ) in (3.74); that is, we will consider $\mathcal{Y}(w_{(1)}, e^\zeta)w_{(2)}$.

Given any $r \in \mathbb{Z}$, we define

$$\Omega_r(\mathcal{Y}) : W_2 \otimes W_1 \rightarrow W_3[\log x]\{x\}$$

by the formula

$$\Omega_r(\mathcal{Y})(w_{(2)}, x)w_{(1)} = e^{xL(-1)}\mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)} \quad (3.75)$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. This expression is indeed well defined because of the truncation condition (3.25) (recall Remark 3.13). The following result generalizes Proposition 7.1 of [HL6] (for the ordinary intertwining operator case) and has essentially the same proof as that proposition, which in turn generalized Proposition 5.4.7 of [FHL]:

Proposition 3.44 *The operator $\Omega_r(\mathcal{Y})$ is a logarithmic intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_2 W_1 \end{smallmatrix}\right)$. Moreover,*

$$\Omega_{-r-1}(\Omega_r(\mathcal{Y})) = \Omega_r(\Omega_{-r-1}(\mathcal{Y})) = \mathcal{Y}. \quad (3.76)$$

In the strongly graded case, if \mathcal{Y} is grading-compatible, then so is $\Omega_r(\mathcal{Y})$, and in particular, the correspondence $\mathcal{Y} \mapsto \Omega_r(\mathcal{Y})$ defines a linear isomorphism from $\mathcal{V}_{W_1 W_2}^{W_3}$ to $\mathcal{V}_{W_2 W_1}^{W_3}$, and we have

$$N_{W_1 W_2}^{W_3} = N_{W_2 W_1}^{W_3}.$$

Proof The lower truncation condition (3.25) is clear. From the Jacobi identity (3.26) for \mathcal{Y} ,

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - y}{x_0} \right) Y_3(v, x_1) \mathcal{Y}(w_{(1)}, y) w_{(2)} \\
& \quad - x_0^{-1} \delta \left(\frac{y - x_1}{-x_0} \right) \mathcal{Y}(w_{(1)}, y) Y_2(v, x_1) w_{(2)} \\
& = y^{-1} \delta \left(\frac{x_1 - x_0}{y} \right) \mathcal{Y}(Y_1(v, x_0) w_{(1)}, y) w_{(2)},
\end{aligned} \tag{3.77}$$

with $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, we obtain

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - y}{x_0} \right) e^{-yL(-1)} Y_3(v, x_1) \mathcal{Y}(w_{(1)}, y) w_{(2)} \Big|_{y^n = e^{(2r+1)\pi i n} x_2^n, (\log y)^k = ((2r+1)\pi i + \log x_2)^k, n \in \mathbb{C}, k \in \mathbb{N}} \\
& \quad - x_0^{-1} \delta \left(\frac{y - x_1}{-x_0} \right) e^{-yL(-1)} \mathcal{Y}(w_{(1)}, y) Y_2(v, x_1) w_{(2)} \Big|_{y^n = e^{(2r+1)\pi i n} x_2^n, (\log y)^k = ((2r+1)\pi i + \log x_2)^k, n \in \mathbb{C}, k \in \mathbb{N}} \\
& = y^{-1} \delta \left(\frac{x_1 - x_0}{y} \right) e^{-yL(-1)} \mathcal{Y}(Y_1(v, x_0) w_{(1)}, y) w_{(2)} \Big|_{y^n = e^{(2r+1)\pi i n} x_2^n, (\log y)^k = ((2r+1)\pi i + \log x_2)^k, n \in \mathbb{C}, k \in \mathbb{N}} \tag{3.78}
\end{aligned}$$

The first term of the left-hand side of (3.78) is equal to

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - y}{x_0} \right) Y_3(v, x_1 - y) e^{-yL(-1)} \mathcal{Y}(w_{(1)}, y) w_{(2)} \Big|_{y^n = e^{(2r+1)\pi i n} x_2^n, (\log y)^k = ((2r+1)\pi i + \log x_2)^k, n \in \mathbb{C}, k \in \mathbb{N}} \\
& = x_0^{-1} \delta \left(\frac{x_1 + x_2}{x_0} \right) Y_3(v, x_0) \Omega_r(\mathcal{Y})(w_{(2)}, x_2) w_{(1)},
\end{aligned}$$

the second term is equal to

$$-x_0^{-1} \delta \left(\frac{-x_2 - x_1}{-x_0} \right) \Omega_r(\mathcal{Y})(Y_2(v, x_1) w_{(2)}, x_2) w_{(1)}$$

and the right-hand side of (3.78) is equal to

$$-x_2^{-1} \delta \left(\frac{x_1 - x_0}{-x_2} \right) \Omega_r(\mathcal{Y})(w_{(2)}, x_2) Y_1(v, x_0) w_{(1)}.$$

Substituting into (3.78) we obtain

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 + x_2}{x_0} \right) Y_3(v, x_0) \Omega_r(\mathcal{Y})(w_{(2)}, x_2) w_{(1)} \\
& \quad - x_0^{-1} \delta \left(\frac{x_2 + x_1}{x_0} \right) \Omega_r(\mathcal{Y})(Y_2(v, x_1) w_{(2)}, x_2) w_{(1)} \\
& = -x_2^{-1} \delta \left(\frac{x_1 - x_0}{-x_2} \right) \Omega_r(\mathcal{Y})(w_{(2)}, x_2) Y_1(v, x_0) w_{(1)},
\end{aligned} \tag{3.79}$$

which is equivalent to

$$\begin{aligned}
& x_1^{-1} \delta \left(\frac{x_0 - x_2}{x_1} \right) Y_3(v, x_0) \Omega_r(\mathcal{Y})(w_{(2)}, x_2) w_{(1)} \\
& \quad - x_1^{-1} \delta \left(\frac{x_2 - x_0}{-x_1} \right) \Omega_r(\mathcal{Y})(w_{(2)}, x_2) Y_1(v, x_0) w_{(1)} \\
& = x_2^{-1} \delta \left(\frac{x_0 - x_1}{x_2} \right) \Omega_r(\mathcal{Y})(Y_2(v, x_1) w_{(2)}, x_2) w_{(1)}
\end{aligned} \tag{3.80}$$

(recall (2.6)). This in turn is the Jacobi identity for $\Omega_r(\mathcal{Y})$ (with the roles of x_0 and x_1 reversed in (3.26)).

To prove the $L(-1)$ -derivative property (3.27) for $\Omega_r(\mathcal{Y})$, first note that from (3.74) and the $L(-1)$ -derivative property for \mathcal{Y} ,

$$\begin{aligned}
\frac{d}{dx} \mathcal{Y}(w_{(1)}, e^\zeta x) w_{(2)} &= e^\zeta \left(\frac{d}{dy} \mathcal{Y}(w_{(1)}, y) w_{(2)} \right) \Big|_{y^n = e^{\zeta n} x^n, (\log y)^k = (\zeta + \log x)^k, n \in \mathbb{C}, k \in \mathbb{N}} \\
&= e^\zeta \mathcal{Y}(L(-1)w_{(1)}, e^\zeta x) w_{(2)},
\end{aligned} \tag{3.81}$$

and in particular,

$$\frac{d}{dx} \mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)} = -\mathcal{Y}(L(-1)w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)}. \tag{3.82}$$

Thus, by using formula (3.28) with $j = -1$ we have

$$\begin{aligned}
\frac{d}{dx} \Omega_r(\mathcal{Y})(w_{(2)}, x) w_{(1)} &= \frac{d}{dx} e^{xL(-1)} \mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)} \\
&= e^{xL(-1)} L(-1) \mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)} + e^{xL(-1)} \frac{d}{dx} \mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)} \\
&= e^{xL(-1)} L(-1) \mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)} - e^{xL(-1)} \mathcal{Y}(L(-1)w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)} \\
&= e^{xL(-1)} \mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i} x) L(-1) w_{(2)} \\
&= \Omega_r(\mathcal{Y})(L(-1)w_{(2)}, x) w_{(1)},
\end{aligned} \tag{3.83}$$

as desired.

In the case that V is Möbius, we prove the $\mathfrak{sl}(2)$ -bracket relations (3.28) for $\Omega_r(\mathcal{Y})$. By using the $\mathfrak{sl}(2)$ -bracket relations for \mathcal{Y} and the relations

$$e^{xL(-1)} L(j) e^{-xL(-1)} = \sum_{i=0}^{j+1} \binom{j+1}{i} (-x)^i L(j-i)$$

for $j = -1, 0$ and -1 (see (3.70)), we have

$$\begin{aligned}
\Omega_r(\mathcal{Y})(L(j)w_{(2)}, x) w_{(1)} &= e^{xL(-1)} \mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i} x) L(j) w_{(2)} = \\
&= e^{xL(-1)} L(j) \mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)}
\end{aligned}$$

$$\begin{aligned}
& -e^{xL(-1)} \sum_{i=0}^{j+1} \binom{j+1}{i} (-x)^i \mathcal{Y}(L(j-i)w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)} \\
& = \sum_{i=0}^{j+1} \binom{j+1}{i} (-x)^i L(j-i) e^{xL(-1)} \mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)} \\
& \quad - e^{xL(-1)} \sum_{i=0}^{j+1} \binom{j+1}{i} (-x)^i \mathcal{Y}(L(j-i)w_{(1)}, e^{(2r+1)\pi i} x) w_{(2)} \\
& = \sum_{i=0}^{j+1} \binom{j+1}{i} (-x)^i (L(j-i) \Omega_r(\mathcal{Y})(w_{(2)}, x) - \Omega_r(\mathcal{Y})(w_{(2)}, x) L(j-i)) w_{(1)},
\end{aligned}$$

which is the alternative form (3.73) of the $\mathfrak{sl}(2)$ -bracket relations for $\Omega_r(\mathcal{Y})$.

The identity (3.76) is clear from the definitions of $\Omega_r(\mathcal{Y})$ and $\Omega_{-r-1}(\mathcal{Y})$, and the remaining assertions are clear. \square

Remark 3.45 For each triple $s_1, s_2, s_3 \in \mathbb{Z}$, the logarithmic intertwining operator \mathcal{Y} gives rise to a logarithmic intertwining operator $\mathcal{Y}_{[s_1, s_2, s_3]}$ of the same type, defined by

$$\mathcal{Y}_{[s_1, s_2, s_3]}(w_{(1)}, x) = e^{2\pi i s_1 L(0)} \mathcal{Y}(e^{2\pi i s_2 L(0)} w_{(1)}, x) e^{2\pi i s_3 L(0)}$$

for $w_{(1)} \in W_1$, by Remarks 3.30 and 3.40. In the strongly graded case, if \mathcal{Y} is grading-compatible, so is $\mathcal{Y}_{[s_1, s_2, s_3]}$. Clearly,

$$\mathcal{Y}_{[0, 0, 0]} = \mathcal{Y}$$

and for $r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{Z}$,

$$(\mathcal{Y}_{[r_1, r_2, r_3]})_{[s_1, s_2, s_3]} = \mathcal{Y}_{[r_1 + s_1, r_2 + s_2, r_3 + s_3]}.$$

For any $a \in \mathbb{C}$, we have the formula

$$e^{aL(0)} \mathcal{Y}(w_{(1)}, x) e^{-aL(0)} = \mathcal{Y}(e^{aL(0)} w_{(1)}, e^a x) \quad (3.84)$$

(cf. (3.59)). This is proved by imitating the proof of (3.59), replacing $y^{L(0)}$ by $e^{aL(0)}$, y by e^a and $\log y$ by a in that proof, using (3.67) in place of (3.50) and keeping in mind formula (3.74). (When (3.35) is used in this proof, for homogeneous elements $w_{(1)}$ and $w_{(2)}$, the exponential series all terminate, as does the sum over $l \in \mathbb{N}$.) From this, we see that (3.76) generalizes to

$$\Omega_s(\Omega_r(\mathcal{Y})) = \mathcal{Y}_{[r+s+1, -(r+s+1), -(r+s+1)]} = \mathcal{Y}(\cdot, e^{2\pi i(r+s+1)} \cdot)$$

for all $r, s \in \mathbb{Z}$.

In case V , W_1 , W_2 and W_3 are strongly graded, which we now assume, we have the concept of “ r -contragredient operator” as follows (in the ordinary intertwining operator case this was introduced in [HL6]): Given a grading-compatible logarithmic intertwining operator

\mathcal{Y} of type $\binom{W_3}{W_1 W_2}$ and an integer r , we define the r -contragredient operator of \mathcal{Y} to be the linear map

$$\begin{aligned} W_1 \otimes W'_3 &\rightarrow W'_2\{x\}[[\log x]] \\ w_{(1)} \otimes w'_{(3)} &\mapsto A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)} \end{aligned}$$

given by

$$\begin{aligned} \langle A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)} \rangle_{W_2} &= \\ = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi i L(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)} \rangle_{W_3}, \end{aligned} \quad (3.85)$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, where we use the notation

$$f(x^{-1}) = \sum_{m \in \mathbb{N}, n \in \mathbb{C}} w_{n,m} x^{-n} (-\log x)^m$$

for any $f(x) = \sum_{m \in \mathbb{N}, n \in \mathbb{C}} w_{n,m} x^n (\log x)^m \in \mathcal{W}\{x\}[[\log x]]$, \mathcal{W} any vector space (not involving x). Note that for the case $W_1 = V$ and $W_2 = W_3 = W$, the operator $A_r(\mathcal{Y})$ agrees with the contragredient vertex operator \mathcal{Y}' (recall (2.57) and (2.73)) for any $r \in \mathbb{Z}$.

We have the following result generalizing Proposition 7.3 in [HL6] for ordinary intertwining operators, and having essentially the same proof (and also generalizing Theorem 5.5.1 and Proposition 5.5.2 of [FHL]):

Proposition 3.46 *The r -contragredient operator $A_r(\mathcal{Y})$ of a grading-compatible logarithmic intertwining operator \mathcal{Y} of type $\binom{W_3}{W_1 W_2}$ is a grading-compatible logarithmic intertwining operator of type $\binom{W'_2}{W_1 W'_3}$. Moreover,*

$$A_{-r-1}(A_r(\mathcal{Y})) = A_r(A_{-r-1}(\mathcal{Y})) = \mathcal{Y}. \quad (3.86)$$

In particular, the correspondence $\mathcal{Y} \mapsto A_r(\mathcal{Y})$ defines a linear isomorphism from $\mathcal{V}_{W_1 W_2}^{W_3}$ to $\mathcal{V}_{W_1 W'_3}^{W'_2}$, and we have

$$N_{W_1 W_2}^{W_3} = N_{W_1 W'_3}^{W'_2}.$$

Proof First we need to show that for $w_{(1)} \in W_1$ and $w'_{(3)} \in W'_3$,

$$A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)} \in W'_2[\log x]\{x\}, \quad (3.87)$$

that is, for each power of x there are only finitely many powers of $\log x$. This and the lower truncation condition (3.25) as well as the grading-compatibility condition (3.30) follow from a variant of the argument proving the lower truncation condition for contragredient vertex operators (recall (2.98)):

Fix elements $w_{(1)} \in W_1$ and $w'_{(3)} \in W'_3$ homogeneous with respect to the double gradings of W_1 and W'_3 (it will suffice to prove the desired assertions for such elements), and in fact take

$$w_{(1)} \in W_1^{(\beta)} \quad \text{and} \quad w'_{(3)} \in (W'_3)^{(\gamma)},$$

where β and γ are elements of the abelian group \tilde{A} , in the notation of Definition 2.25, and fix $n \in \mathbb{C}$. The right-hand side of (3.85) is a (finite) sum of terms of the form

$$\langle w'_{(3)}, \mathcal{Y}(w, x^{-1})w_{(2)} \rangle x^p (\log x)^q \quad (3.88)$$

where $w \in W_1$ is doubly homogeneous and in fact $w \in W_1^{(\beta)}$ (by (2.88)), and where $p \in \mathbb{C}$ and $q \in \mathbb{N}$. (The pairing $\langle \cdot, \cdot \rangle$ is between W'_3 and W_3 .) Let $w_{(2)}^{(-\beta-\gamma)}$ be the component of (the arbitrary element) $w_{(2)}$ in $W_2^{(-\beta-\gamma)}$, with respect to the \tilde{A} -grading. Then (3.88) equals

$$\langle w'_{(3)}, \mathcal{Y}(w, x^{-1})w_{(2)}^{-\beta-\gamma} \rangle x^p (\log x)^q \quad (3.89)$$

because of the grading-compatibility condition (3.30) together with (2.94). (This is why we need our logarithmic intertwining operators to be grading-compatible.) This shows in particular that

$$w_{(1)N;K}^{\mathcal{Y}} w'_{(3)} \in (W'_2)^{(\beta+\gamma)} \quad (3.90)$$

for $N \in \mathbb{C}$ and $K \in \mathbb{N}$, so that (3.30) holds for $A_r(\mathcal{Y})$. Let us write (3.89) as

$$\sum_{l \in \mathbb{C}} \sum_{k \in \mathbb{N}} \langle w'_{(3)}, w_{l;k}^{\mathcal{Y}} w_{(2)}^{-\beta-\gamma} \rangle x^{l+1+p} (\log x)^{k+q} = \sum_{m \in \mathbb{C}} \sum_{k \in \mathbb{N}} \langle w'_{(3)}, w_{n-p-1-m;k}^{\mathcal{Y}} w_{(2)}^{-\beta-\gamma} \rangle x^{n-m} (\log x)^{k+q} \quad (3.91)$$

(recall that we have fixed $n \in \mathbb{C}$). But each term $\langle w'_{(3)}, w_{n-p-1-m;k}^{\mathcal{Y}} w_{(2)}^{-\beta-\gamma} \rangle$ in (3.91) can be replaced by

$$\langle w'_{(3)}, w_{n-p-1-m;k}^{\mathcal{Y}} u^{[m]} \rangle, \quad (3.92)$$

where $u^{[m]} \in W_3^{(-\beta-\gamma)}$ is the component of $w_{(2)}^{-\beta-\gamma}$, with respect to the generalized-weight grading, of (generalized) weight

$$\text{wt } u^{[m]} = \text{wt } w'_{(3)} - \text{wt } w + n - p - m,$$

by Proposition 3.21(b). To see that the coefficient of x^n in (3.91) involves only finitely many powers of $\log x$, independently of the element $w_{(2)}$, we take $m = 0$ in (3.91) and we observe that the possible elements $u^{[0]}$ range through the space

$$(W_3)_{[\text{wt } w'_{(3)} - \text{wt } w + n - p]}^{(-\beta-\gamma)},$$

which is finite dimensional by the grading restriction condition (2.86). This proves (3.87). To prove the lower truncation condition (3.25), what we must show is that for sufficiently large $m \in \mathbb{N}$, the coefficient of x^{n-m} in (3.91) is 0 (independently of $w_{(2)}$). But by the grading restriction condition (2.85),

$$(W_3)_{[\text{wt } w'_{(3)} - \text{wt } w + n - p - m]}^{(-\beta-\gamma)} = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large.}$$

Hence the coefficient of x^{n-m} in (3.91) is zero for $m \in \mathbb{N}$ sufficiently large, as desired, proving the lower truncation condition.

For the Jacobi identity, we need to show that

$$\begin{aligned}
& \left\langle x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_2'(v, x_1) A_r(\mathcal{Y})(w_{(1)}, x_2) w'_{(3)}, w_{(2)} \right\rangle_{W_2} \\
& - \left\langle x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) A_r(\mathcal{Y})(w_{(1)}, x_2) Y_3'(v, x_1) w'_{(3)}, w_{(2)} \right\rangle_{W_2} \\
& = \left\langle x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) A_r(\mathcal{Y})(Y_1(v, x_0) w_{(1)}, x_2) w'_{(3)}, w_{(2)} \right\rangle_{W_2}. \tag{3.93}
\end{aligned}$$

By the definitions (2.73) and (3.85) we have

$$\begin{aligned}
& \langle Y_2'(v, x_1) A_r(\mathcal{Y})(w_{(1)}, x_2) w'_{(3)}, w_{(2)} \rangle_{W_2} \\
& = \langle w'_{(3)}, \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) \cdot \\
& \quad \cdot Y_2(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_1^{-1}) w_{(2)} \rangle_{W_3}, \tag{3.94}
\end{aligned}$$

$$\begin{aligned}
& \langle A_r(\mathcal{Y})(w_{(1)}, x_2) Y_3'(v, x_1) w'_{(3)}, w_{(2)} \rangle_{W_2} \\
& = \langle w'_{(3)}, Y_3(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_1^{-1}) \cdot \\
& \quad \cdot \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) w_{(2)} \rangle_{W_3}, \tag{3.95}
\end{aligned}$$

$$\begin{aligned}
& \langle A_r(\mathcal{Y})(Y_1(v, x_0) w_{(1)}, x_2) w'_{(3)}, w_{(2)} \rangle_{W_2} \\
& = \langle w'_{(3)}, \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 Y_1(v, x_0) w_{(1)}, x_2^{-1}) w_{(2)} \rangle_{W_3}. \tag{3.96}
\end{aligned}$$

From the Jacobi identity for \mathcal{Y} we have

$$\begin{aligned}
& \left\langle w'_{(3)}, \left(\frac{-x_0}{x_1 x_2} \right)^{-1} \delta \left(\frac{x_1^{-1} - x_2^{-1}}{-x_0/x_1 x_2} \right) Y_3(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_1^{-1}) \cdot \right. \\
& \quad \left. \cdot \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) w_{(2)} \right\rangle_{W_3} \\
& - \left\langle w'_{(3)}, \left(\frac{-x_0}{x_1 x_2} \right)^{-1} \delta \left(\frac{x_2^{-1} - x_1^{-1}}{x_0/x_1 x_2} \right) \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) \cdot \right. \\
& \quad \left. \cdot Y_2(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_1^{-1}) w_{(2)} \right\rangle_{W_3} \\
& = \left\langle w'_{(3)}, (x_2^{-1})^{-1} \delta \left(\frac{x_1^{-1} + x_0/x_1 x_2}{x_2^{-1}} \right) \mathcal{Y}(Y_1(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, -x_0/x_1 x_2) \cdot \right. \\
& \quad \left. \cdot e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) w_{(2)} \right\rangle_{W_3}, \tag{3.97}
\end{aligned}$$

or equivalently,

$$\begin{aligned}
& - \left\langle w'_{(3)}, x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_3(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_1^{-1}) \cdot \right. \\
& \quad \left. \cdot \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) w_{(2)} \right\rangle_{W_3} \\
& + \left\langle w'_{(3)}, x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) \cdot \right. \\
& \quad \left. \cdot Y_2(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_1^{-1}) w_{(2)} \right\rangle_{W_3} \\
& = \left\langle w'_{(3)}, x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) \mathcal{Y}(Y_1(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, -x_0/x_1 x_2) \cdot \right. \\
& \quad \left. \cdot e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}) w_{(2)} \right\rangle_{W_3}. \tag{3.98}
\end{aligned}$$

Substituting (3.94), (3.95) and (3.96) into (3.93) and then comparing with (3.98), we see that the proof of (3.93) is reduced to the proof of the formula

$$\begin{aligned}
& x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 Y_1(v, x_0) w_{(1)}, x_2^{-1}) \\
& = x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) \mathcal{Y}(Y_1(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, -x_0/x_1 x_2) \cdot \\
& \quad \cdot e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}), \tag{3.99}
\end{aligned}$$

or of

$$\begin{aligned}
& \mathcal{Y}(e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 Y_1(v, x_0) w_{(1)}, x_2^{-1}) \\
& = \mathcal{Y}(Y_1(e^{(x_2+x_0)L(1)} (-(x_2+x_0)^{-2})^{L(0)} v, -x_0/(x_2+x_0)x_2) \cdot \\
& \quad \cdot e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 w_{(1)}, x_2^{-1}). \tag{3.100}
\end{aligned}$$

We see that we need only prove

$$\begin{aligned}
& e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 Y_1(v, x_0) \\
& = Y_1(e^{(x_2+x_0)L(1)} (-(x_2+x_0)^{-2})^{L(0)} v, -x_0/(x_2+x_0)x_2) \cdot \\
& \quad \cdot e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 \tag{3.101}
\end{aligned}$$

or equivalently, the conjugation formula

$$\begin{aligned}
& e^{x_2 L(1)} e^{(2r+1)\pi i L(0)} (x_2^{-L(0)})^2 Y_1(v, x_0) (x_2^{L(0)})^2 e^{-(2r+1)\pi i L(0)} e^{-x_2 L(1)} \\
& = Y_1(e^{(x_2+x_0)L(1)} (-(x_2+x_0)^{-2})^{L(0)} v, -x_0/(x_2+x_0)x_2) \tag{3.102}
\end{aligned}$$

for $v \in V$, acting on the module W_1 . But formula (3.102) follows from (3.59), (3.60) and the formula

$$e^{(2r+1)\pi i L(0)} Y_1(v, x) e^{-(2r+1)\pi i L(0)} = Y_1((-1)^{L(0)} v, -x), \quad (3.103)$$

which is a special case of (3.84). This establishes the Jacobi identity.

The $L(-1)$ -derivative property follows from the same argument used in the proof of Theorem 5.5.1 of [FHL]: We have (omitting the subscript W_3 on the pairings after a certain point)

$$\begin{aligned} \left\langle \frac{d}{dx} A_r(\mathcal{Y})(w_{(1)}, x) w'_{(3)}, w_{(2)} \right\rangle_{W_2} &= \frac{d}{dx} \langle A_r(\mathcal{Y})(w_{(1)}, x) w'_{(3)}, w_{(2)} \rangle_{W_2} \\ &= \frac{d}{dx} \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)} e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 w_{(1)}, x^{-1}) w_{(2)} \rangle_{W_3} \\ &= \langle w'_{(3)}, \frac{d}{dx} \mathcal{Y}(e^{xL(1)} e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 w_{(1)}, x^{-1}) w_{(2)} \rangle \\ &= \langle w'_{(3)}, \mathcal{Y}(\frac{d}{dx} (e^{xL(1)} e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 w_{(1)}), x^{-1}) w_{(2)} \rangle \\ &\quad + \langle w'_{(3)}, \frac{d}{dx} \mathcal{Y}(w, x^{-1})|_{w=e^{xL(1)} e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 w_{(1)}} w_{(2)} \rangle \\ &= \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)} L(1) e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 w_{(1)}, x^{-1}) w_{(2)} \rangle \\ &\quad + \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)} (-2L(0)x^{-1}) e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 w_{(1)}, x^{-1}) w_{(2)} \rangle \\ &\quad + \langle w'_{(3)}, \frac{d}{dx^{-1}} \mathcal{Y}(w, x^{-1})|_{w=e^{xL(1)} e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 w_{(1)}} w_{(2)} \rangle (-x^{-2}) \\ &= \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)} (2xL(0) - x^2 L(1)) e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 (-x^{-2}) w_{(1)}, x^{-1}) w_{(2)} \rangle \\ &\quad + \langle w'_{(3)}, \mathcal{Y}(L(-1) e^{xL(1)} e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 w_{(1)}, x^{-1}) w_{(2)} \rangle (-x^{-2}) \\ &= \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)} (2xL(0) - x^2 L(1)) e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 (-x^{-2}) w_{(1)}, x^{-1}) w_{(2)} \rangle \\ &\quad + \langle w'_{(3)}, \mathcal{Y}(L(-1) e^{xL(1)} e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 (-x^{-2}) w_{(1)}, x^{-1}) w_{(2)} \rangle \end{aligned} \quad (3.104)$$

Now by (3.72), with x replaced by $-x$, we have

$$L(-1) e^{xL(1)} = e^{xL(1)} (L(-1) - 2xL(0) + x^2 L(1)).$$

Using this together with (3.64) and (3.71) (with x specialized to $-(2r+1)\pi i$, and the convergence of the exponential series invoked; recall Remark 3.39), we see that the right-hand side of (3.104) equals

$$\begin{aligned} &\langle w'_{(3)}, \mathcal{Y}(e^{xL(1)} L(-1) e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 (-x^{-2}) w_{(1)}, x^{-1}) w_{(2)} \rangle \\ &= \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)} e^{(2r+1)\pi i L(0)} (-L(-1)) (x^{-L(0)})^2 (-x^{-2}) w_{(1)}, x^{-1}) w_{(2)} \rangle \\ &= \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)} e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 L(-1) w_{(1)}, x^{-1}) w_{(2)} \rangle \\ &= \langle A_r(\mathcal{Y})(L(-1) w_{(1)}, x) w'_{(3)}, w_{(2)} \rangle_{W_2}, \end{aligned}$$

as desired.

We now show that, in case V is Möbius, the $\mathfrak{sl}(2)$ -bracket relations (3.28) hold for $A_r(\mathcal{Y})$. For these, we first see that, for $j = -1, 0, 1$, by using (2.75), (3.85) and the $\mathfrak{sl}(2)$ -bracket relations (3.28) for \mathcal{Y} we have

$$\begin{aligned}
& \langle L'(j)A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)} \rangle_{W_2} \\
&= \langle A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, L(-j)w_{(2)} \rangle_{W_2} \\
&= \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})L(-j)w_{(2)} \rangle_{W_3} \\
&= \langle w'_{(3)}, L(-j)\mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})w_{(2)} \rangle_{W_3} \\
&\quad - \left\langle w'_{(3)}, \sum_{i=0}^{-j+1} \binom{-j+1}{i} x^{-i} \cdot \right. \\
&\quad \left. \cdot \mathcal{Y}(L(-j-i)e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})w_{(2)} \right\rangle_{W_3} \quad (3.105)
\end{aligned}$$

Now from (3.72), (3.71) (with x specialized to $-(2r+1)\pi i$) and (3.64), one computes that

$$\begin{aligned}
& (x^{L(0)})^2 e^{-(2r+1)\pi iL(0)} e^{-xL(1)} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix} e^{xL(1)} e^{(2r+1)\pi iL(0)} (x^{-L(0)})^2 \\
&= \begin{pmatrix} -x^2 & -2x & -1 \\ 0 & 1 & x^{-1} \\ 0 & 0 & -x^{-2} \end{pmatrix} \begin{pmatrix} L(-1) \\ L(0) \\ L(1) \end{pmatrix}
\end{aligned}$$

on W_1 , which implies that

$$\begin{aligned}
& \sum_{i=0}^{-j+1} \binom{-j+1}{i} x^{-i} L(-j-i) e^{xL(1)} e^{(2r+1)\pi iL(0)} (x^{-L(0)})^2 \\
&= - \sum_{i=0}^{j+1} \binom{j+1}{i} x^i e^{xL(1)} e^{(2r+1)\pi iL(0)} (x^{-L(0)})^2 L(j-i)
\end{aligned}$$

on W_1 . Hence the right-hand side of (3.105) is equal to

$$\begin{aligned}
& \langle L'(j)w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})w_{(2)} \rangle_{W_3} \\
&+ \sum_{i=0}^{j+1} \binom{j+1}{i} x^i \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi iL(0)}(x^{-L(0)})^2L(j-i)w_{(1)}, x^{-1})w_{(2)} \rangle_{W_2} \\
&= \langle A_r(\mathcal{Y})(w_{(1)}, x)L'(j)w'_{(3)}, w_{(2)} \rangle_{W_2} \\
&+ \sum_{i=0}^{j+1} \binom{j+1}{i} x^i \langle A_r(\mathcal{Y})(L(j-i)w_{(1)}, x)w'_{(3)}, w_{(2)} \rangle_{W_2},
\end{aligned}$$

and the $\mathfrak{sl}(2)$ -bracket relations for $A_r(\mathcal{Y})$ are proved. (Note that this argument essentially generalizes the proof of Lemma 2.22.) We have finished proving that $A_r(\mathcal{Y})$ is a grading-compatible logarithmic intertwining operator.

Finally, for the relation (3.86), we of course identify W_2'' with W_2 and W_3'' with W_3 , according to Theorem 2.34. Let us view \mathcal{Y} as a grading-compatible logarithmic intertwining operator of type $\left(\begin{smallmatrix} W_2' \\ W_1 W_3' \end{smallmatrix}\right)$, so that $A_r(\mathcal{Y})$ is such an operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right)$. We have

$$\begin{aligned} & \langle A_{-r-1}A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)} \rangle_{W_2} \\ &= \langle w'_{(3)}, A_r(\mathcal{Y})(e^{xL(1)}e^{(-2r-1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x^{-1})w_{(2)} \rangle_{W_3} \\ &= \langle \mathcal{Y}(e^{x^{-1}L(1)}e^{(2r+1)\pi iL(0)}(x^{L(0)})^2e^{xL(1)}e^{(-2r-1)\pi iL(0)}(x^{-L(0)})^2w_{(1)}, x)w'_{(3)}, w_{(2)} \rangle_{W_2} \\ &= \langle \mathcal{Y}(w_{(1)}, x)w'_{(3)}, w_{(2)} \rangle_{W_2}, \end{aligned}$$

where the last equality is due to the relation

$$e^{(2r+1)\pi iL(0)}(x^{L(0)})^2e^{xL(1)}(x^{-L(0)})^2e^{-(2r+1)\pi iL(0)} = e^{-x^{-1}L(1)} \quad (3.106)$$

on W_1 , whose proof is similar to that of formula (5.3.1) of [FHL]. Namely, (3.106) follows from the relation

$$e^{(2r+1)\pi iL(0)}(x^{L(0)})^2xL(1)(x^{-L(0)})^2e^{-(2r+1)\pi iL(0)} = -x^{-1}L(1), \quad (3.107)$$

which realizes the transformation $x \mapsto -\frac{1}{x}$, and (3.107) follows from (3.64) together with (3.71) specialized as above. \square

Remark 3.47 The last argument in the proof shows that for any $r, s \in \mathbb{Z}$, formula (3.86) generalizes to:

$$A_s(A_r(\mathcal{Y})) = \mathcal{Y}_{[0, r+s+1, 0]}$$

(recall Remark 3.45).

With V , W_1 , W_2 and W_3 strongly graded, set

$$N_{W_1W_2W_3} = N_{W_1W_2}^{W_3'}. \quad (3.108)$$

Then Proposition 3.44 gives

$$N_{W_1W_2W_3} = N_{W_2W_1W_3}$$

and Proposition 3.46 gives

$$N_{W_1W_2W_3} = N_{W_1W_3W_2}.$$

Thus for any permutation σ of $(1, 2, 3)$,

$$N_{W_1W_2W_3} = N_{W_{\sigma(1)}W_{\sigma(2)}W_{\sigma(3)}}. \quad (3.109)$$

It is clear from Proposition 3.21(b) that in the nontrivial logarithmic intertwining operator case, taking projections of $\mathcal{Y}(w_{(1)}, x)w_{(2)}$ to (generalized) weight subspaces is not enough to recover its coefficients of $x^n(\log x)^k$ for $n \in \mathbb{C}$ and $k \in \mathbb{N}$, in contrast with the (ordinary) intertwining operator case (cf. [HL5], the paragraph containing formula (4.17)). However, taking projections of certain related intertwining operators does indeed suffice for this purpose:

Proposition 3.48 *Let W_1, W_2, W_3 be generalized modules for a Möbius (or conformal) vertex algebra V and let \mathcal{Y} be a logarithmic intertwining operator of type $\binom{W_3}{W_1 W_2}$. Let $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ be homogeneous of generalized weights n_1 and n_2 , respectively. Then for any $n \in \mathbb{C}$ and any $r \in \mathbb{N}$, $w_{(1)}^{\mathcal{Y}}_{n;r} w_{(2)}$ can be written as a certain linear combination of products of the component of weight $n_1 + n_2 - n - 1$ of*

$$(L(0) - n_1 - n_2 + n + 1)^l \mathcal{Y}((L(0) - n_1)^i w_{(1)}, x) (L(0) - n_2)^j w_{(2)}$$

for certain $i, j, l \in \mathbb{N}$ with monomials of the form $x^{n+1}(\log x)^m$ for certain $m \in \mathbb{N}$.

Proof Multiplying (3.49) by $x^{-n-1}(\log x)^k$ and summing over $k \in \mathbb{N}$ (a finite sum by definition) we have that for any $t \in \mathbb{N}$,

$$\begin{aligned} & \sum_{k \in \mathbb{N}} \binom{k+t}{t} w_{(1)}^{\mathcal{Y}}_{n; k+t} w_{(2)} x^{-n-1} (\log x)^k \left(= x^{-n-1} \sum_{k \in \mathbb{N}} \binom{k}{t} w_{(1)}^{\mathcal{Y}}_{n; k} w_{(2)} (\log x)^{k-t} \right) \\ &= \sum_{i, j, l \in \mathbb{N}, i+j+l=t} \frac{1}{i!j!l!} (-1)^{i+j} \sum_{k \in \mathbb{N}} (L(0) - n_1 - n_2 + n + 1)^l \cdot \\ & \quad \cdot ((L(0) - n_1)^i w_{(1)})^{\mathcal{Y}}_{n; k} (L(0) - n_2)^j w_{(2)} x^{-n-1} (\log x)^k \\ &= \sum_{i, j, l \in \mathbb{N}, i+j+l=t} \frac{1}{i!j!l!} (-1)^{i+j} \pi_{n_1+n_2-n-1} ((L(0) - n_1 - n_2 + n + 1)^l \cdot \\ & \quad \cdot \mathcal{Y}((L(0) - n_1)^i w_{(1)}, x) (L(0) - n_2)^j w_{(2)})). \end{aligned} \quad (3.110)$$

Let K be a positive integer such that $w_{(1)}^{\mathcal{Y}}_{n; k'} w_{(2)} = 0$ for all $k' \geq K$. Denote the right-hand side of (3.110) by $\pi(t, w_{(1)}, w_{(2)}, x, \log x)$. Then by putting the identities (3.110) for $t = 0, 1, \dots, K-1$ together in matrix form we have

$$x^{-n-1} A \begin{pmatrix} w_{(1)}^{\mathcal{Y}}_{n; 0} w_{(2)} \\ w_{(1)}^{\mathcal{Y}}_{n; 1} w_{(2)} \\ \vdots \\ w_{(1)}^{\mathcal{Y}}_{n; K-1} w_{(2)} \end{pmatrix} = \begin{pmatrix} \pi(0, w_{(1)}, w_{(2)}, x, \log x) \\ \pi(1, w_{(1)}, w_{(2)}, x, \log x) \\ \vdots \\ \pi(K-1, w_{(1)}, w_{(2)}, x, \log x) \end{pmatrix} \quad (3.111)$$

where A is the $K \times K$ matrix whose (i, j) -entry is equal to $\binom{j-1}{i-1} (\log x)^{j-i}$. Letting P_K be the triangular matrix whose (i, j) -entry is $\binom{j-1}{i-1}$ (an upper triangular ‘‘Pascal matrix’’), we have

$$A = \text{diag}(1, (\log x)^{-1}, \dots, (\log x)^{-(K-1)}) \cdot P_K \cdot \text{diag}(1, \log x, \dots, (\log x)^{K-1}).$$

Its inverse is

$$A^{-1} = \text{diag}(1, (\log x)^{-1}, \dots, (\log x)^{-(K-1)}) \cdot P_K^{-1} \cdot \text{diag}(1, \log x, \dots, (\log x)^{K-1})$$

and the (i, j) -entry of P_K^{-1} is $(-1)^{i+j} \binom{j-1}{i-1}$. Now multiplying the left-hand side of (3.111) by $x^{n+1}A^{-1}$ we obtain

$$\begin{pmatrix} w_{(1)n;0}^{\mathcal{Y}} w_{(2)} \\ w_{(1)n;1}^{\mathcal{Y}} w_{(2)} \\ \vdots \\ w_{(1)n;K-1}^{\mathcal{Y}} w_{(2)} \end{pmatrix} = x^{n+1}A^{-1} \begin{pmatrix} \pi(0, w_{(1)}, w_{(2)}, x, \log x) \\ \pi(1, w_{(1)}, w_{(2)}, x, \log x) \\ \vdots \\ \pi(K-1, w_{(1)}, w_{(2)}, x, \log x) \end{pmatrix}$$

or explicitly,

$$(w_{(1)})_{n;r}^{\mathcal{Y}} w_{(2)} = x^{n+1} \sum_{t=r}^{K-1} (-1)^{r+t} \binom{t}{r} (\log x)^{t-r} \pi(t, w_{(1)}, w_{(2)}, x, \log x) \quad (3.112)$$

for $r = 0, 1, \dots, K-1$. (In particular, all x 's and $\log x$'s cancel out in the right-hand side of (3.112).) \square

4 The notions of $P(z)$ - and $Q(z)$ -tensor product

We now generalize to the setting of the present work the notions of $P(z)$ - and $Q(z)$ -tensor product of modules introduced in [HL5], [HL6] and [HL7]. We introduce the notions of $P(z)$ - and $Q(z)$ -intertwining map among strongly \tilde{A} -graded generalized modules for a strongly \tilde{A} -graded Möbius or conformal vertex algebra V and establish the relationship between such intertwining maps and grading-compatible logarithmic intertwining operators. We define the $P(z)$ - and $Q(z)$ -tensor products of two strongly \tilde{A} -graded generalized V -modules using these intertwining maps and natural universal properties. As examples, for a strongly \tilde{A} -graded generalized module W , we construct and describe the $P(z)$ -tensor products of V and W and also of W and V ; the underlying strongly \tilde{A} -graded generalized modules of the tensor product structures are W itself, in both of these cases. In the case in which V is a finitely reductive vertex operator algebra (recall the Introduction), we construct and describe the $P(z)$ - and $Q(z)$ -tensor products of arbitrary V -modules, and we use this structure to motivate the construction of associativity isomorphisms that we will carry out in later sections.

In view of the results in Sections 2 and 3 involving contragredient modules, it is natural for us to work in the strongly-graded setting from now on:

Assumption 4.1 *Throughout this section and the remainder of this work, we shall assume the following, unless other assumptions are explicitly made: A is an abelian group and \tilde{A} is an abelian group containing A as a subgroup; V is a strongly A -graded Möbius or conformal vertex algebra; all V -modules and generalized V -modules considered are strongly \tilde{A} -graded; and all intertwining operators and logarithmic intertwining operators considered are grading-compatible. (Recall Definitions 2.23, 2.25, 3.11 and 3.15.) Also, in this section, z will be a fixed nonzero complex number.*

4.1 The notion of $P(z)$ -tensor product

We first generalize the notion of $P(z)$ -intertwining map given in Section 4 of [HL5]; our $P(z)$ -intertwining maps will automatically be grading-compatible by definition. We use the notations given in Definition 2.18. The main part of the following definition, the Jacobi identity (4.4), was previewed in the Introduction (formula (1.19)). It should be compared with the corresponding formula (1.1) in the Lie algebra setting, and with the Jacobi identity (3.26) in the definition of the notion of logarithmic intertwining operator; note that the formal variable x_2 in that Jacobi identity is specialized here to the nonzero complex number z . Also, the $\mathfrak{sl}(2)$ -bracket relations (4.5) should be compared with the corresponding relations (3.28). There is no $L(-1)$ -derivative formula for intertwining maps; as we shall see, the $P(z)$ -intertwining maps are obtained from logarithmic intertwining operators by a process of specialization of the formal variable to the complex variable z .

Definition 4.2 Let (W_1, Y_1) , (W_2, Y_2) and (W_3, Y_3) be generalized V -modules. A $P(z)$ -intertwining map of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ is a linear map

$$I : W_1 \otimes W_2 \rightarrow \overline{W}_3 \quad (4.1)$$

(recall from Definition 2.18 that \overline{W}_3 is the formal completion of W_3 with respect to the \mathbb{C} -grading) such that the following conditions are satisfied: the *grading compatibility condition*: for $\beta, \gamma \in \tilde{A}$ and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$,

$$I(w_{(1)} \otimes w_{(2)}) \in \overline{W_3^{(\beta+\gamma)}}; \quad (4.2)$$

the *lower truncation condition*: for any elements $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and any $n \in \mathbb{C}$,

$$\pi_{n-m} I(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large} \quad (4.3)$$

(which follows from (4.2), in view of the grading restriction condition (2.85); recall the notation π_n from Definition 2.18); the *Jacobi identity*:

$$\begin{aligned} x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) Y_3(v, x_1) I(w_{(1)} \otimes w_{(2)}) \\ = z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) I(Y_1(v, x_0) w_{(1)} \otimes w_{(2)}) \\ + x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) I(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \end{aligned} \quad (4.4)$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that all the expressions in the right-hand side of (4.4) are well defined, and that the left-hand side of (4.4) is meaningful because any infinite linear combination of v_n ($n \in \mathbb{Z}$) of the form $\sum_{n < N} a_n v_n$ ($a_n \in \mathbb{C}$) acts in a well-defined way on any $I(w_{(1)} \otimes w_{(2)})$, in view of (4.3)); and the $\mathfrak{sl}(2)$ -bracket relations: for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$L(j) I(w_{(1)} \otimes w_{(2)}) = I(w_{(1)} \otimes L(j) w_{(2)}) + \sum_{i=0}^{j+1} \binom{j+1}{i} z^i I((L(j-i) w_{(1)}) \otimes w_{(2)}) \quad (4.5)$$

for $j = -1, 0$ and 1 (note that if V is in fact a conformal vertex algebra, this follows automatically from (4.4) by setting $v = \omega$ and taking $\text{Res}_{x_0} \text{Res}_{x_1} x_1^{j+1}$). The vector space of $P(z)$ -intertwining maps of type $\binom{W_3}{W_1 W_2}$ is denoted by $\mathcal{M}[P(z)]_{W_1 W_2}^{W_3}$, or simply by $\mathcal{M}_{W_1 W_2}^{W_3}$ if there is no ambiguity.

Remark 4.3 As we mentioned in the Introduction, $P(z)$ is the Riemann sphere $\hat{\mathbb{C}}$ with one negatively oriented puncture at ∞ and two ordered positively oriented punctures at z and 0 , with local coordinates $1/w$, $w - z$ and w , respectively, vanishing at these three punctures. The geometry underlying the notion of $P(z)$ -intertwining map and the notions of $P(z)$ -product and $P(z)$ -tensor product (see below) is determined by $P(z)$.

Remark 4.4 In the case of \mathbb{C} -graded ordinary modules for a vertex operator algebra, where the grading restriction condition (2.89) for a module W is replaced by the (more restrictive) condition

$$W_{(n)} = 0 \quad \text{for } n \in \mathbb{C} \text{ with sufficiently negative real part} \quad (4.6)$$

as in [HL5] (and where, in our context, the abelian groups A and \tilde{A} are trivial), the notion of $P(z)$ -intertwining map above agrees with the earlier one introduced in [HL5]; in this case, the conditions (4.2) and (4.3) are automatic.

Remark 4.5 As in Remark 3.42, it is clear that the $\mathfrak{sl}(2)$ -bracket relations (4.5) can equivalently be written as

$$\begin{aligned} I(L(j)w_{(1)} \otimes w_{(2)}) &= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i L(j-i) I(w_{(1)} \otimes w_{(2)}) \\ &\quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i I(w_{(1)} \otimes L(j-i)w_{(2)}) \end{aligned} \quad (4.7)$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $j = -1, 0$ and 1 .

Following [HL5] we will choose the branch of $\log z$ so that

$$0 \leq \operatorname{Im}(\log z) < 2\pi \quad (4.8)$$

(despite the fact that we happened to have used a different branch in (3.12) in the proof of Theorem 3.6). We will also use the notation

$$l_p(z) = \log z + 2\pi ip, \quad p \in \mathbb{Z}, \quad (4.9)$$

as in [HL5], for arbitrary values of the log function. For a formal expression $f(x)$ as in (3.2), but involving only nonnegative integral powers of $\log x$, and $\zeta \in \mathbb{C}$, whenever

$$f(x) \Big|_{x^n=e^{\zeta n}, (\log x)^m=\zeta^m, n \in \mathbb{C}, m \in \mathbb{N}} \quad (4.10)$$

exists algebraically, we will write (4.10) simply as $f(x) \Big|_{x=e^\zeta}$ or $f(e^\zeta)$, and we will call this “substituting e^ζ for x in $f(x)$,” even though, in general, it depends on ζ , not just on e^ζ . (See also (3.74).) In addition, for a fixed integer p , we will sometimes write

$$f(x) \Big|_{x=z} \quad \text{or} \quad f(z) \quad (4.11)$$

instead of $f(x) \Big|_{x=e^{l_p(z)}}$ or $f(e^{l_p(z)})$. We will sometimes say that “ $f(e^\zeta)$ exists” or that “ $f(z)$ exists.”

Remark 4.6 A very important example of an $f(z)$ existing in this sense occurs when $f(x) = \mathcal{Y}(w_{(1)}, x)w_{(2)} (\in W_3[\log x]\{x\})$ for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and a logarithmic intertwining operator \mathcal{Y} of type $\binom{W_3}{W_1 W_2}$, in the notation of Definition 3.11; note that (4.10) exists (as an element of $\overline{W_3}$) in this case because of Proposition 3.21(b). Note also that in particular, $\mathcal{Y}(w_{(1)}, e^\zeta)$ (or $\mathcal{Y}(w_{(1)}, z)$) exists as a linear map from W_2 to $\overline{W_3}$, and that $\mathcal{Y}(\cdot, z) \cdot$ exists as a linear map

$$\begin{aligned} W_1 \otimes W_2 &\rightarrow \overline{W_3} \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, z)w_{(2)}. \end{aligned} \quad (4.12)$$

Now we use these considerations to construct correspondences between (grading-compatible) logarithmic intertwining operators and $P(z)$ -intertwining maps. Fix an integer p . Let \mathcal{Y} be a logarithmic intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then we have a linear map

$$I_{\mathcal{Y},p} : W_1 \otimes W_2 \rightarrow \overline{W}_3 \quad (4.13)$$

defined by

$$I_{\mathcal{Y},p}(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, e^{l_p(z)})w_{(2)} \quad (4.14)$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. The grading-compatibility condition (3.30) yields the grading-compatibility condition (4.2) for $I_{\mathcal{Y},p}$, and (4.3) follows. By substituting $e^{l_p(z)}$ for x_2 in (3.26) and for x in (3.28), we see that $I_{\mathcal{Y},p}$ satisfies the Jacobi identity (4.4) and the $\mathfrak{sl}(2)$ -bracket relations (4.5). Hence $I_{\mathcal{Y},p}$ is a $P(z)$ -intertwining map.

On the other hand, we note that (3.59) is equivalent to

$$\langle y^{L'(0)}w'_{(3)}, \mathcal{Y}(y^{-L(0)}w_{(1)}, x)y^{-L(0)}w_{(2)} \rangle_{W_3} = \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, xy)w_{(2)} \rangle_{W_3} \quad (4.15)$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, where we are using the pairing between the contragredient module W'_3 and W_3 or \overline{W}_3 (recall Definition 2.32, Theorem 2.34, (2.75), (2.98) and (3.53)). Substituting $e^{l_p(z)}$ for x and then $e^{-l_p(z)}x$ for y , we obtain

$$\begin{aligned} & \langle y^{L'(0)}x^{L'(0)}w'_{(3)}, \mathcal{Y}(y^{-L(0)}x^{-L(0)}w_{(1)}, e^{l_p(z)})y^{-L(0)}x^{-L(0)}w_{(2)} \rangle_{W_3} \Big|_{y=e^{-l_p(z)}} \\ &= \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle_{W_3}, \end{aligned}$$

or equivalently, using the notation (4.14),

$$\begin{aligned} & \langle w'_{(3)}, y^{L(0)}x^{L(0)}I_{\mathcal{Y},p}(y^{-L(0)}x^{-L(0)}w_{(1)} \otimes y^{-L(0)}x^{-L(0)}w_{(2)}) \rangle_{W_3} \Big|_{y=e^{-l_p(z)}} \\ &= \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \rangle_{W_3}. \end{aligned}$$

Thus we have recovered \mathcal{Y} from $I_{\mathcal{Y},p}$.

This motivates the following definition: Given a $P(z)$ -intertwining map I and an integer p , we define a linear map

$$\mathcal{Y}_{I,p} : W_1 \otimes W_2 \rightarrow W_3[\log x]\{x\} \quad (4.16)$$

by

$$\begin{aligned} & \mathcal{Y}_{I,p}(w_{(1)}, x)w_{(2)} \\ &= y^{L(0)}x^{L(0)}I(y^{-L(0)}x^{-L(0)}w_{(1)} \otimes y^{-L(0)}x^{-L(0)}w_{(2)}) \Big|_{y=e^{-l_p(z)}} \end{aligned} \quad (4.17)$$

for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (this is well defined and indeed maps to $W_3[\log x]\{x\}$, in view of (3.53)). We will also use the notation $w_{(1)n;k}^{I,p}w_{(2)} \in W_3$ defined by

$$\mathcal{Y}_{I,p}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} w_{(1)n;k}^{I,p}w_{(2)}x^{-n-1}(\log x)^k. \quad (4.18)$$

Observe that since the operator $x^{\pm L(0)}$ always increases the power of x in an expression homogeneous of generalized weight n by $\pm n$, we see from (4.17) that

$$w_{(1)n;k}^{I,p} w_{(2)} \in (W_3)_{[n_1+n_2-n-1]} \quad (4.19)$$

for $w_{(1)} \in (W_1)_{[n_1]}$ and $w_{(2)} \in (W_2)_{[n_2]}$. Moreover, for $I = I_{\mathcal{Y},p}$, we have $\mathcal{Y}_{I,p} = \mathcal{Y}$, and for $\mathcal{Y} = \mathcal{Y}_{I,p}$, we have $I_{\mathcal{Y},p} = I$.

We can now prove the following proposition generalizing Proposition 12.2 in [HL7].

Proposition 4.7 *For $p \in \mathbb{Z}$, the correspondence $\mathcal{Y} \mapsto I_{\mathcal{Y},p}$ is a linear isomorphism from the space $\mathcal{V}_{W_1 W_2}^{W_3}$ of (grading-compatible) logarithmic intertwining operators of type $(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix})$ to the space $\mathcal{M}_{W_1 W_2}^{W_3}$ of $P(z)$ -intertwining maps of the same type. Its inverse map is given by $I \mapsto \mathcal{Y}_{I,p}$.*

Proof We need only show that for any $P(z)$ -intertwining map I of type $(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix})$, $\mathcal{Y}_{I,p}$ is a logarithmic intertwining operator of the same type. The lower truncation condition (4.3) implies that the lower truncation condition (3.25) for logarithmic intertwining operator holds for $\mathcal{Y}_{I,p}$. Let us now prove the Jacobi identity for $\mathcal{Y}_{I,p}$.

Changing the formal variables x_0 and x_1 to $x_0 e^{l_p(z)} x_2^{-1}$ and $x_1 e^{l_p(z)} x_2^{-1}$, respectively, in the Jacobi identity (4.4) for I , and then changing v to $y^{-L(0)} x_2^{-L(0)} v \Big|_{y=e^{-l_p(z)}}$ we obtain (noting that at first, $e^{l_p(z)}$ could be written simply as z because only integral powers occur)

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_3(y^{-L(0)} x_2^{-L(0)} v, x_1 y^{-1} x_2^{-1}) I(w_{(1)} \otimes w_{(2)}) \Big|_{y=e^{-l_p(z)}} \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) I(Y_1(y^{-L(0)} x_2^{-L(0)} v, x_0 y^{-1} x_2^{-1}) w_{(1)} \otimes w_{(2)}) \Big|_{y=e^{-l_p(z)}} \\ & \quad + x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) I(w_{(1)} \otimes Y_2(y^{-L(0)} x_2^{-L(0)} v, x_1 y^{-1} x_2^{-1}) w_{(2)}) \Big|_{y=e^{-l_p(z)}}. \end{aligned}$$

Using the formula

$$Y_3(y^{-L(0)} x_2^{-L(0)} v, x_1 y^{-1} x_2^{-1}) = y^{-L(0)} x_2^{-L(0)} Y_3(v, x_1) y^{L(0)} x_2^{L(0)},$$

which holds on the generalized module W_3 , by (3.59), and the similar formulas for Y_1 and Y_2 , we get

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) y^{-L(0)} x_2^{-L(0)} Y_3(v, x_1) y^{L(0)} x_2^{L(0)} I(w_{(1)} \otimes w_{(2)}) \Big|_{y=e^{-l_p(z)}} \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) I(y^{-L(0)} x_2^{-L(0)} Y_1(v, x_0) y^{L(0)} x_2^{L(0)} w_{(1)} \otimes w_{(2)}) \Big|_{y=e^{-l_p(z)}} \\ & \quad + x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) I(w_{(1)} \otimes y^{-L(0)} x_2^{-L(0)} Y_2(v, x_1) y^{L(0)} x_2^{L(0)} w_{(2)}) \Big|_{y=e^{-l_p(z)}}. \end{aligned}$$

Replacing $w_{(1)}$ by $y^{-L(0)}x_2^{-L(0)}w_{(1)}\Big|_{y=e^{-l_p(z)}}$ and $w_{(2)}$ by $y^{-L(0)}x_2^{-L(0)}w_{(2)}\Big|_{y=e^{-l_p(z)}}$, and then applying $y^{L(0)}x_2^{L(0)}\Big|_{y=e^{-l_p(z)}}$ to the whole equation, we obtain

$$\begin{aligned}
& x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_3(v, x_1)y^{L(0)}x_2^{L(0)} \\
& \quad \cdot I(y^{-L(0)}x_2^{-L(0)}w_{(1)} \otimes y^{-L(0)}x_2^{-L(0)}w_{(2)})\Big|_{y=e^{-l_p(z)}} \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)y^{L(0)}x_2^{L(0)} \\
& \quad \cdot I(y^{-L(0)}x_2^{-L(0)}Y_1(v, x_0)w_{(1)} \otimes y^{-L(0)}x_2^{-L(0)}w_{(2)})\Big|_{y=e^{-l_p(z)}} \\
& \quad + x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)y^{L(0)}x_2^{L(0)} \\
& \quad \cdot I(y^{-L(0)}x_2^{-L(0)}w_{(1)} \otimes y^{-L(0)}x_2^{-L(0)}Y_2(v, x_1)w_{(2)})\Big|_{y=e^{-l_p(z)}}.
\end{aligned}$$

But using (4.17), we can write this as

$$\begin{aligned}
& x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_3(v, x_1)\mathcal{Y}_{I,p}(w_{(1)}, x_2)w_{(2)} \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}_{I,p}(Y_1(v, x_0)w_{(1)}, x_2)w_{(2)} \\
& \quad + x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}_{I,p}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)}.
\end{aligned}$$

That is, the Jacobi identity for $\mathcal{Y}_{I,p}$ holds.

Similar procedures show that the $\mathfrak{sl}(2)$ -bracket relations for I imply the $\mathfrak{sl}(2)$ -bracket relations for $\mathcal{Y}_{I,p}$, as follows: Let j be $-1, 0$ or 1 . By multiplying (4.5) by $(yx)^j$ and using (3.64) we obtain

$$\begin{aligned}
& (yx)^{-L(0)}L(j)(yx)^{L(0)}I(w_{(1)} \otimes w_{(2)}) \\
& = I(w_{(1)} \otimes (yx)^{-L(0)}L(j)(yx)^{L(0)}w_{(2)}) \\
& \quad + \sum_{i=0}^{j+1} \binom{j+1}{i} z^i (yx)^i I(((yx)^{-L(0)}L(j-i)(yx)^{L(0)}w_{(1)}) \otimes w_{(2)})
\end{aligned}$$

Replacing $w_{(1)}$ by $(yx)^{-L(0)}w_{(1)}$ and $w_{(2)}$ by $(yx)^{-L(0)}w_{(2)}$, and then applying $(yx)^{L(0)}$ to the whole equation, we obtain

$$\begin{aligned}
& L(j)(yx)^{L(0)}I((yx)^{-L(0)}w_{(1)} \otimes (yx)^{-L(0)}w_{(2)}) \\
& = (yx)^{L(0)}I((yx)^{-L(0)}w_{(1)} \otimes (yx)^{-L(0)}L(j)w_{(2)}) \\
& \quad + \sum_{i=0}^{j+1} \binom{j+1}{i} z^i (yx)^i (yx)^{L(0)}I(((yx)^{-L(0)}L(j-i)w_{(1)}) \otimes (yx)^{-L(0)}w_{(2)}).
\end{aligned}$$

Evaluating at $y = e^{-l_p(z)}$ and using (4.17) we see that this gives exactly the $\mathfrak{sl}(2)$ -bracket relations (3.28) for $\mathcal{Y}_{I,p}$.

Finally, we prove the $L(-1)$ -derivative property for $\mathcal{Y}_{I,p}$. This follows from (4.17), (3.55), and the $\mathfrak{sl}(2)$ -bracket relation with $j = 0$ for $\mathcal{Y}_{I,p}$, namely,

$$[L(0), \mathcal{Y}_{I,p}(w_{(1)}, x)] = \mathcal{Y}_{I,p}(L(0)w_{(1)}, x) + x\mathcal{Y}_{I,p}(L(-1)w_{(1)}, x),$$

as follows:

$$\begin{aligned} & \frac{d}{dx} \mathcal{Y}_{I,p}(w_{(1)}, x)w_{(2)} \\ &= \frac{d}{dx} e^{-l_p(z)L(0)} x^{L(0)} I(e^{l_p(z)L(0)} x^{-L(0)} w_{(1)} \otimes e^{l_p(z)L(0)} x^{-L(0)} w_{(2)}) \\ &= e^{-l_p(z)L(0)} x^{-1} x^{L(0)} L(0) I(e^{l_p(z)L(0)} x^{-L(0)} w_{(1)} \otimes e^{l_p(z)L(0)} x^{-L(0)} w_{(2)}) \\ &\quad - e^{-l_p(z)L(0)} x^{L(0)} I(e^{l_p(z)L(0)} x^{-1} x^{-L(0)} L(0) w_{(1)} \otimes e^{l_p(z)L(0)} x^{-L(0)} w_{(2)}) \\ &\quad - e^{-l_p(z)L(0)} x^{L(0)} I(e^{l_p(z)L(0)} x^{-L(0)} w_{(1)} \otimes e^{l_p(z)L(0)} x^{-1} x^{-L(0)} L(0) w_{(2)}) \\ &= x^{-1} L(0) \mathcal{Y}_{I,p}(w_{(1)}, x)w_{(2)} - x^{-1} \mathcal{Y}_{I,p}(w_{(1)}, x) L(0)w_{(2)} \\ &\quad - x^{-1} \mathcal{Y}_{I,p}(L(0)w_{(1)}, x)w_{(2)} \\ &= \mathcal{Y}_{I,p}(L(-1)w_{(1)}, x)w_{(2)}. \end{aligned} \quad \square$$

Remark 4.8 Given a generalized V -module (W, Y_W) , recall from Remark 3.17 that Y_W is a logarithmic intertwining operator of type $\binom{W}{VW}$ not involving $\log x$ and having only integral powers of x . Then the substitution $x \mapsto z$ in (4.14) is very simple; it is independent of p and $Y_W(\cdot, z) \cdot$ entails only the substitutions $x^n \mapsto z^n$ for $n \in \mathbb{Z}$. As a special case, we can take (W, Y_W) to be (V, Y) itself.

Remark 4.9 Let I be a $P(z)$ -intertwining map of type $\binom{W_3}{W_1 W_2}$ and let $p, p' \in \mathbb{Z}$. From (4.17), we see that the logarithmic intertwining operators $\mathcal{Y}_{I,p}$ and $\mathcal{Y}_{I,p'}$ of this same type differ as follows:

$$\begin{aligned} & \mathcal{Y}_{I,p'}(w_{(1)}, x)w_{(2)} \\ &= e^{2\pi i(p-p')L(0)} \mathcal{Y}_{I,p}(e^{2\pi i(p'-p)L(0)} w_{(1)}, x) e^{2\pi i(p'-p)L(0)} w_{(2)} \end{aligned} \quad (4.20)$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Using the notation in Remark 3.45, we thus have

$$\begin{aligned} \mathcal{Y}_{I,p'} &= (\mathcal{Y}_{I,p})_{[p-p', p'-p, p'-p]} \\ &= \mathcal{Y}_{I,p}(\cdot, e^{2\pi i(p'-p)} \cdot). \end{aligned} \quad (4.21)$$

Remark 4.10 Let I be a $P(z)$ -intertwining map of type $\binom{W_3}{W_1 W_2}$. Then from the correspondence between $P(z)$ -intertwining maps and logarithmic intertwining operators in Proposition 4.7, we see that for any nonzero complex number z_1 , the linear map I_1 defined by

$$I_1(w_{(1)} \otimes w_{(2)}) = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} w_{(1)}^{I,p}_{n,k} w_{(2)} e^{l_p(z_1)(-n-1)} (l_p(z_1))^k \quad (4.22)$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (recall (4.18)) is a $P(z_1)$ -intertwining map of the same type. In this sense, $w_{(1)n;k}^{I,p} w_{(2)}$ is independent of z . We will hence sometimes write $I(w_{(1)} \otimes w_{(2)})$ as

$$I(w_{(1)}, z)w_{(2)}, \quad (4.23)$$

indicating that z can be replaced by any nonzero complex number. However, for a general intertwining map associated to a sphere with punctures not necessarily of type $P(z)$, the corresponding element $w_{(1)n;k}^{I,p} w_{(2)}$ will in general be different.

We now proceed to the definition of the $P(z)$ -tensor product. As in [HL5], this will be a suitably universal “ $P(z)$ -product.” We generalize these notions from [HL5] using the notations \mathcal{M}_{sg} and \mathcal{GM}_{sg} (the categories of strongly graded V -modules and generalized V -modules, respectively; recall Notation 2.36) as follows:

Definition 4.11 Let \mathcal{C}_1 be either of the categories \mathcal{M}_{sg} or \mathcal{GM}_{sg} . For $W_1, W_2 \in \text{ob } \mathcal{C}_1$, a $P(z)$ -product of W_1 and W_2 is an object (W_3, Y_3) of \mathcal{C}_1 equipped with a $P(z)$ -intertwining map I_3 of type $\binom{W_3}{W_1 W_2}$. We denote it by $(W_3, Y_3; I_3)$ or simply by $(W_3; I_3)$. Let $(W_4, Y_4; I_4)$ be another $P(z)$ -product of W_1 and W_2 . A *morphism* from $(W_3, Y_3; I_3)$ to $(W_4, Y_4; I_4)$ is a module map η from W_3 to W_4 such that the diagram

$$\begin{array}{ccc} & W_1 \otimes W_2 & \\ I_3 \swarrow & & \searrow I_4 \\ \overline{W}_3 & \xrightarrow{\bar{\eta}} & \overline{W}_4 \end{array}$$

commutes, that is,

$$I_4 = \bar{\eta} \circ I_3, \quad (4.24)$$

where

$$\bar{\eta} : \overline{W}_3 \rightarrow \overline{W}_4 \quad (4.25)$$

is the natural map uniquely extending η . (Note that $\bar{\eta}$ exists because η preserves \mathbb{C} -gradings; we shall use the notation $\bar{\eta}$ for any such map η .)

Remark 4.12 In this setting, let η be a morphism from $(W_3, Y_3; I_3)$ to $(W_4, Y_4; I_4)$. We know from (4.16)–(4.18) that for $p \in \mathbb{Z}$, the coefficients $w_{(1)n;k}^{I_3,p} w_{(2)}$ and $w_{(1)n;k}^{I_4,p} w_{(2)}$ in the formal expansion (4.18) of $\mathcal{Y}_{I_3,p}(w_{(1)}, x)w_{(2)}$ and $\mathcal{Y}_{I_4,p}(w_{(1)}, x)w_{(2)}$, respectively, are determined by I_3 and I_4 , and that

$$\eta(w_{(1)n;k}^{I_3,p} w_{(2)}) = w_{(1)n;k}^{I_4,p} w_{(2)}, \quad (4.26)$$

as we see by applying $\bar{\eta}$ to (4.17).

The notion of $P(z)$ -tensor product is now defined by means of a universal property as follows:

Definition 4.13 Let \mathcal{C} be a full subcategory of either \mathcal{M}_{sg} or \mathcal{GM}_{sg} . For $W_1, W_2 \in \text{ob } \mathcal{C}$, a $P(z)$ -tensor product of W_1 and W_2 in \mathcal{C} is a $P(z)$ -product $(W_0, Y_0; I_0)$ with $W_0 \in \text{ob } \mathcal{C}$ such that for any $P(z)$ -product $(W, Y; I)$ with $W \in \text{ob } \mathcal{C}$, there is a unique morphism from $(W_0, Y_0; I_0)$ to $(W, Y; I)$. Clearly, a $P(z)$ -tensor product of W_1 and W_2 in \mathcal{C} , if it exists, is unique up to unique isomorphism. In this case we will denote it by

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$$

and call the object $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ the $P(z)$ -tensor product module of W_1 and W_2 in \mathcal{C} . We will skip the phrase “in \mathcal{C} ” if the category \mathcal{C} under consideration is clear in context.

Remark 4.14 Consider the functor from \mathcal{C} to the category **Set** defined by assigning to $W \in \text{ob } \mathcal{C}$ the set $\mathcal{M}_{W_1 W_2}^W$ of all $P(z)$ -intertwining maps of type $\binom{W}{W_1 W_2}$. Then if the $P(z)$ -tensor product of W_1 and W_2 exists, it is just the universal element for this functor, and this functor is representable, represented by the $P(z)$ -tensor product. (Recall that given a functor f from a category \mathcal{K} to **Set**, a universal element for f , if it exists, is a pair (X, x) where $X \in \text{ob } \mathcal{K}$ and $x \in f(X)$ such that for any pair (Y, y) with $Y \in \text{ob } \mathcal{K}$ and $y \in f(Y)$, there is a unique morphism $\sigma : X \rightarrow Y$ such that $f(\sigma)(x) = y$; in this case, f is represented by X .)

Definition 4.13 and Proposition 4.7 immediately give the following result relating the module maps from a $P(z)$ -tensor product module with the $P(z)$ -intertwining maps and the logarithmic intertwining operators:

Proposition 4.15 Suppose that $W_1 \boxtimes_{P(z)} W_2$ exists. We have a natural isomorphism

$$\begin{aligned} \text{Hom}_V(W_1 \boxtimes_{P(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{M}_{W_1 W_2}^{W_3} \\ \eta &\mapsto \bar{\eta} \circ \boxtimes_{P(z)} \end{aligned} \quad (4.27)$$

and for $p \in \mathbb{Z}$, a natural isomorphism

$$\begin{aligned} \text{Hom}_V(W_1 \boxtimes_{P(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{V}_{W_1 W_2}^{W_3} \\ \eta &\mapsto \mathcal{Y}_{\eta, p} \end{aligned} \quad (4.28)$$

where $\mathcal{Y}_{\eta, p} = \mathcal{Y}_{I, p}$ with $I = \bar{\eta} \circ \boxtimes_{P(z)}$. \square

Suppose that the $P(z)$ -tensor product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ of W_1 and W_2 exists. We will sometimes denote the action of the canonical $P(z)$ -intertwining map

$$w_{(1)} \otimes w_{(2)} \mapsto \boxtimes_{P(z)}(w_{(1)} \otimes w_{(2)}) = \boxtimes_{P(z)}(w_{(1)}, z)w_{(2)} \in \overline{W_1 \boxtimes_{P(z)} W_2} \quad (4.29)$$

(recall (4.23)) on elements simply by $w_{(1)} \boxtimes_{P(z)} w_{(2)}$:

$$w_{(1)} \boxtimes_{P(z)} w_{(2)} = \boxtimes_{P(z)}(w_{(1)} \otimes w_{(2)}) = \boxtimes_{P(z)}(w_{(1)}, z)w_{(2)}. \quad (4.30)$$

Remark 4.16 We emphasize that the element $w_{(1)} \boxtimes_{P(z)} w_{(2)}$ defined here is an element of the formal completion $\overline{W_1 \boxtimes_{P(z)} W_2}$, and *not* (in general) of the module $W_1 \boxtimes_{P(z)} W_2$ itself. This is different from the classical case for modules for a Lie algebra (recall Section 1.1), where the tensor product of elements of two modules is an element of the tensor product module.

Remark 4.17 Note that under the natural isomorphism (4.27) for the case $W_3 = W_1 \boxtimes_{P(z)} W_2$, the identity map from $W_1 \boxtimes_{P(z)} W_2$ to itself corresponds to the canonical intertwining map $\boxtimes_{P(z)}$. Furthermore, for $p \in \mathbb{Z}$, the $P(z)$ -tensor product of W_1 and W_2 gives rise to a logarithmic intertwining operator $\mathcal{Y}_{\boxtimes_{P(z)}, p}$ of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$, according to formula (4.17). If p is changed to $p' \in \mathbb{Z}$, this logarithmic intertwining operator changes according to (4.20). Note that the $P(z)$ -intertwining map $\boxtimes_{P(z)}$ is canonical and depends only on z , while a corresponding logarithmic intertwining operator is not; it depends on $p \in \mathbb{Z}$.

Remark 4.18 Sometimes it will be convenient, as in the next proposition, to use the particular isomorphism associated with $p = 0$ (in Proposition 4.7) between the spaces of $P(z)$ -intertwining maps and of logarithmic intertwining operators of the same type. In this case, we shall sometimes simplify the notation by dropping the $p (= 0)$ in the notation $w_{(1)n;k}^{I,0} w_{(2)}$ (recall 4.18)):

$$w_{(1)n;k}^I w_{(2)} = w_{(1)n;k}^{I,0} w_{(2)}. \quad (4.31)$$

Proposition 4.19 *Suppose that the $P(z)$ -tensor product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ of W_1 and W_2 in \mathcal{C} exists. Then for any complex number $z_1 \neq 0$, the $P(z_1)$ -tensor product of W_1 and W_2 in \mathcal{C} also exists, and is given by $(W_1 \boxtimes_{P(z_1)} W_2, Y_{P(z_1)}; \boxtimes_{P(z_1)})$, where the $P(z_1)$ -intertwining map $\boxtimes_{P(z_1)}$ is defined by*

$$\boxtimes_{P(z_1)} (w_{(1)} \otimes w_{(2)}) = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} w_{(1)n;k}^{\boxtimes_{P(z)}} w_{(2)} e^{\log z_1 (-n-1)} (\log z_1)^k \quad (4.32)$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

Proof By Remark 4.10, (4.32) indeed defines a $P(z_1)$ -product. Given any $P(z_1)$ -product $(W_3, Y_3; I_1)$ of W_1 and W_2 , let I be the $P(z)$ -product related to I_1 by formula (4.22) with I_1 , I and z_1 in (4.22) replaced by I , I_1 and z , respectively. Then from the definition of $P(z)$ -tensor product, there is a unique morphism η from $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ to $(W_3, Y_3; I)$. Thus by (4.30) and (4.32) we see that η is also a morphism from the $P(z_1)$ -product $(W_1 \boxtimes_{P(z_1)} W_2, Y_{P(z_1)}; \boxtimes_{P(z_1)})$ to $(W_3, Y_3; I_1)$. The uniqueness of such a morphism follows similarly from the uniqueness of a morphism from $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ to $(W_3, Y_3; I)$. Hence $(W_1 \boxtimes_{P(z_1)} W_2, Y_{P(z_1)}; \boxtimes_{P(z_1)})$ is the $P(z_1)$ -tensor product of W_1 and W_2 . \square

In general, it will turn out that the existence of tensor product, and the tensor product module itself, do not depend on the geometric data. It is the intertwining map from the two modules to the completion of their tensor product that encodes the geometric information.

Generalizing Lemma 4.9 of [H1], we have:

Proposition 4.20 *The module $W_1 \boxtimes_{P(z)} W_2$ (if it exists) is spanned (as a vector space) by the (generalized-) weight components of the elements of $\overline{W_1 \boxtimes_{P(z)} W_2}$ of the form $w_{(1)} \boxtimes_{P(z)} w_{(2)}$, for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.*

Proof Denote by W_0 the vector subspace of $W_1 \boxtimes_{P(z)} W_2$ spanned by all the weight components of all the elements of $\overline{W_1 \boxtimes_{P(z)} W_2}$ of the form $w_{(1)} \boxtimes_{P(z)} w_{(2)}$ for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. For a homogeneous vector $v \in V$ and arbitrary elements $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, equating the $x_0^{-1}x_1^{-m-1}$ coefficients of the Jacobi identity (4.4) gives

$$v_m(w_{(1)} \boxtimes_{P(z)} w_{(2)}) = w_{(1)} \boxtimes_{P(z)} (v_m w_{(2)}) + \sum_{i \in \mathbb{N}} \binom{m}{i} z^{m-i} (v_i w_{(1)}) \boxtimes_{P(z)} w_{(2)} \quad (4.33)$$

for all $m \in \mathbb{Z}$. Note that the summation in the right-hand side of (4.33) is always finite. Hence by taking arbitrary weight components of (4.33) we see that W_0 is closed under the action of V . In case V is Möbius, a similar argument, using (4.5), shows that W_0 is stable under the action of $\mathfrak{sl}(2)$. It is clear that W_0 is \mathbb{C} -graded and \tilde{A} -graded. Thus W_0 is a submodule of $W_1 \boxtimes_{P(z)} W_2$. Now consider the quotient module

$$W = (W_1 \boxtimes_{P(z)} W_2) / W_0$$

and let π_W be the canonical map from $W_1 \boxtimes_{P(z)} W_2$ to W . By the definition of W_0 , we have

$$\pi_W \circ \boxtimes_{P(z)} = 0,$$

using the notation (4.29). The universal property of the $P(z)$ -tensor product then demands that $\pi_W = 0$, i.e., that $W_0 = W_1 \boxtimes_{P(z)} W_2$. \square

It is clear from Definition 4.13 that the tensor product operation distributes over direct sums in the following sense:

Proposition 4.21 *For $U_1, \dots, U_k, W_1, \dots, W_l \in \text{ob } \mathcal{C}$, suppose that each $U_i \boxtimes_{P(z)} W_j$ exists. Then $(\coprod_i U_i) \boxtimes_{P(z)} (\coprod_j W_j)$ exists and there is a natural isomorphism*

$$\left(\coprod_i U_i \right) \boxtimes_{P(z)} \left(\coprod_j W_j \right) \xrightarrow{\sim} \coprod_{i,j} U_i \boxtimes_{P(z)} W_j. \quad \square$$

Remark 4.22 It is of course natural to view the $P(z)$ -tensor product as a bifunctor: Suppose that \mathcal{C} is a full subcategory of either \mathcal{M}_{sg} or \mathcal{GM}_{sg} (recall Notation 2.36) such that for all $W_1, W_2 \in \text{ob } \mathcal{C}$, the $P(z)$ -tensor product of W_1 and W_2 exists in \mathcal{C} . Then $\boxtimes_{P(z)}$ provides a (bi)functor

$$\boxtimes_{P(z)} : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \quad (4.34)$$

as follows: For $W_1, W_2 \in \text{ob } \mathcal{C}$,

$$\boxtimes_{P(z)} (W_1, W_2) = W_1 \boxtimes_{P(z)} W_2 \in \text{ob } \mathcal{C} \quad (4.35)$$

and for V -module maps

$$\sigma_1 : W_1 \rightarrow W_3, \quad (4.36)$$

$$\sigma_2 : W_2 \rightarrow W_4 \quad (4.37)$$

with $W_3, W_4 \in \text{ob } \mathcal{C}$, we have the V -module map, denoted

$$\boxtimes_{P(z)} (\sigma_1, \sigma_2) = \sigma_1 \boxtimes_{P(z)} \sigma_2, \quad (4.38)$$

from $W_1 \boxtimes_{P(z)} W_2$ to $W_3 \boxtimes_{P(z)} W_4$, defined by the universal property of the $P(z)$ -tensor product $W_1 \boxtimes_{P(z)} W_2$ and the fact that the composition of $\boxtimes_{P(z)}$ with $\sigma_1 \otimes \sigma_2$ is a $P(z)$ -intertwining map

$$\boxtimes_{P(z)} \circ (\sigma_1 \otimes \sigma_2) : W_1 \otimes W_2 \rightarrow \overline{W_3 \boxtimes_{P(z)} W_4}. \quad (4.39)$$

Note that it is the effect of this bifunctor on morphisms (rather than on objects) that exhibits the role of the geometric data.

We now discuss the simplest examples of $P(z)$ -tensor products—those in which one or both of W_1 or W_2 is V itself (viewed as a (generalized) V -module); we suppose here that $V \in \text{ob } \mathcal{C}$. Since the discussion of the case in which both W_1 and W_2 are V turns out to be no simpler than the case in which $W_1 = V$, we shall discuss only the two more general cases $W_1 = V$ and $W_2 = V$.

Example 4.23 Let (W, Y_W) be an object of \mathcal{C} . The vertex operator map Y_W gives a $P(z)$ -intertwining map

$$I_{Y_W, p} = Y_W(\cdot, z) \cdot : V \otimes W \rightarrow \overline{W}$$

for any fixed $p \in \mathbb{Z}$ (recall Proposition 4.7 and Remark 4.8). We claim that $(W, Y_W; Y_W(\cdot, z) \cdot)$ is the $P(z)$ -tensor product of V and W in \mathcal{C} . In fact, let $(W_3, Y_3; I)$ be a $P(z)$ -product of V and W in \mathcal{C} and suppose that there exists a module map $\eta : W \rightarrow W_3$ such that

$$\overline{\eta} \circ (Y_W(\cdot, z) \cdot) = I. \quad (4.40)$$

Then for $w \in W$, we must have

$$\begin{aligned} \eta(w) &= \eta(Y_W(\mathbf{1}, z)w) \\ &= (\overline{\eta} \circ (Y_W(\cdot, z) \cdot))(\mathbf{1} \otimes w) \\ &= I(\mathbf{1} \otimes w), \end{aligned} \quad (4.41)$$

so that η is unique if it exists. We now define $\eta : W \rightarrow \overline{W_3}$ using (4.41). We shall show that $\eta(W) \subset W_3$ and that η has the desired properties. Since I is a $P(z)$ -intertwining map of type $\binom{W_3}{VW}$, it corresponds to a logarithmic intertwining operator $\mathcal{Y} = \mathcal{Y}_{I, p}$ of the same type, according to Proposition 4.7. Since $L(-1)\mathbf{1} = 0$, we have

$$\frac{d}{dx} \mathcal{Y}(\mathbf{1}, x) = \mathcal{Y}(L(-1)\mathbf{1}, x) = 0.$$

Thus $\mathcal{Y}(\mathbf{1}, x)$ is simply the constant map $\mathbf{1}_{-1;0}^{\mathcal{Y}} : W \rightarrow W_3$ (using the notation (3.24)), and this map preserves (generalized) weights, by Proposition 3.21(b). By Proposition 4.7, $I = I_{\mathcal{Y},p}$, so that

$$\begin{aligned}\eta(w) &= I(\mathbf{1} \otimes w) \\ &= I_{\mathcal{Y},p}(\mathbf{1} \otimes w) \\ &= \mathbf{1}_{-1;0}^{\mathcal{Y}} w\end{aligned}$$

for $w \in W$. So $\eta = \mathbf{1}_{-1;0}^{\mathcal{Y}}$ is a linear map from W to W_3 preserving (generalized) weights. Using the Jacobi identity for the $P(z)$ -intertwining map I and the fact that $Y(u, x_0)\mathbf{1} \in V[[x_0]]$ for $u \in V$, we obtain

$$\begin{aligned}\eta(Y_W(u, x)w) &= I(\mathbf{1} \otimes Y_W(u, x)w) \\ &= Y_3(u, x)I(\mathbf{1} \otimes w) - \text{Res}_{x_0} z^{-1} \delta\left(\frac{x - x_0}{z}\right) I(Y(u, x_0)\mathbf{1} \otimes w) \\ &= Y_3(u, x)I(\mathbf{1} \otimes w) \\ &= Y_3(u, x)\eta(w)\end{aligned}$$

for $u \in V$ and $w \in W$, proving that η is a module map. For $w \in W$,

$$\begin{aligned}(\bar{\eta} \circ (Y_W(\cdot, z) \cdot))(\mathbf{1} \otimes w) &= \bar{\eta}(Y_W(\mathbf{1}, z)w) \\ &= \eta(w) \\ &= I(\mathbf{1} \otimes w).\end{aligned}\tag{4.42}$$

Using the Jacobi identity for $P(z)$ -intertwining maps, we obtain

$$\begin{aligned}I(Y(u, x_0)v \otimes w) \\ = \text{Res}_x x_0^{-1} \delta\left(\frac{x - z}{x_0}\right) Y_3(u, x)I(v \otimes w) - \text{Res}_x x_0^{-1} \delta\left(\frac{z - x}{-x_0}\right) I(v \otimes Y_W(u, x)w)\end{aligned}\tag{4.43}$$

for $u, v \in V$ and $w \in W$. Since η is a module map and $Y_W(\cdot, z) \cdot$ is a $P(z)$ -intertwining map of type $\binom{W}{VW}$, $\bar{\eta} \circ Y_W(\cdot, z) \cdot$ is a $P(z)$ -intertwining map of type $\binom{W_3}{VW}$. In particular, (4.43) holds when we replace I by $\bar{\eta} \circ Y_W(\cdot, z) \cdot$. Using (4.43) for $v = \mathbf{1}$ together with (4.42), we obtain

$$(\bar{\eta} \circ (Y_W(\cdot, z) \cdot))(u \otimes w) = I(u \otimes w)$$

for $u \in V$ and $w \in W$, proving (4.40), as desired. Thus $(W, Y_W; Y_W(\cdot, z) \cdot)$ is the $P(z)$ -tensor product of V and W in \mathcal{C} .

Example 4.24 Let (W, Y_W) be an object of \mathcal{C} . In order to construct the $P(z)$ -tensor product $W \boxtimes_{P(z)} V$, recall from (3.75) and Proposition 3.44 that $\Omega_p(Y_W)$ is a logarithmic intertwining operator of type $\binom{W}{WV}$. It involves only integral powers of the formal variable and no logarithms, and it is independent of p . In fact,

$$\Omega_p(Y_W)(w, x)v = e^{xL(-1)}Y_W(v, -x)w$$

for $v \in V$ and $w \in W$. For $q \in \mathbb{Z}$,

$$I_{\Omega_p(Y_W),q} = \Omega_p(Y_W)(\cdot, z) \cdot : W \otimes V \rightarrow \overline{W}$$

is a $P(z)$ -intertwining map of the same type and is independent of q . We claim that $(W, Y_W; \Omega_p(Y_W)(\cdot, z) \cdot)$ is the $P(z)$ -tensor product of W and V in \mathcal{C} . In fact, let $(W_3, Y_3; I)$ be a $P(z)$ -product of W and V in \mathcal{C} and suppose that there exists a module map $\eta : W \rightarrow W_3$ such that

$$\overline{\eta} \circ \Omega_p(Y_W)(\cdot, z) \cdot = I. \quad (4.44)$$

For $w \in W$, we must have

$$\begin{aligned} \eta(w) &= \eta(Y_W(\mathbf{1}, -z)w) \\ &= e^{-zL(-1)} \overline{\eta}(e^{zL(-1)} Y_W(\mathbf{1}, -z)w) \\ &= e^{-zL(-1)} \overline{\eta}(\Omega_p(Y_W)(w, z)\mathbf{1}) \\ &= e^{-zL(-1)} (\overline{\eta} \circ (\Omega_p(Y_W)(\cdot, z) \cdot))(w \otimes \mathbf{1}) \\ &= e^{-zL(-1)} I(w \otimes \mathbf{1}), \end{aligned} \quad (4.45)$$

and so η is unique if it exists. We now define $\eta : W \rightarrow \overline{W}_3$ by (4.45). Consider the logarithmic intertwining operator $\mathcal{Y} = \mathcal{Y}_{I,q}$ that corresponds to I by Proposition 4.7. Using Proposition 4.7, (4.8)–(4.10), (3.74) and the equality

$$\begin{aligned} l_q(-z) &= \log |-z| + i(\arg(-z) + 2\pi q) \\ &= \begin{cases} \log |z| + i(\arg z + \pi + 2\pi q), & 0 \leq \arg z < \pi \\ \log |z| + i(\arg z - \pi + 2\pi q), & \pi \leq \arg z < 2\pi \end{cases} \\ &= \begin{cases} l_q(z) + \pi i, & 0 \leq \arg z < \pi \\ l_q(z) - \pi i, & \pi \leq \arg z < 2\pi, \end{cases} \end{aligned}$$

we have

$$\begin{aligned} e^{-zL(-1)} I(w \otimes \mathbf{1}) &= e^{-zL(-1)} \mathcal{Y}(w, e^{l_q(z)} \mathbf{1}) \\ &= e^{-xL(-1)} \mathcal{Y}(w, x) \mathbf{1} \big|_{x^n = e^{nl_q(z)}, (\log x)^m = (l_q(z))^m, n \in \mathbb{C}, m \in \mathbb{N}} \\ &= e^{yL(-1)} \mathcal{Y}(w, e^{\pm \pi i} y) \mathbf{1} \big|_{y^n = e^{nl_q(-z)}, (\log y)^m = (l_q(-z))^m, n \in \mathbb{C}, m \in \mathbb{N}}, \end{aligned}$$

where $e^{\pm \pi i}$ is $e^{-\pi i}$ when $0 \leq \arg z < \pi$ and is $e^{\pi i}$ when $\pi \leq \arg z < 2\pi$. Then by (3.75), we see that $\eta(w) = e^{-zL(-1)} I(w \otimes \mathbf{1})$ is equal to $\Omega_{-1}(\mathcal{Y})(\mathbf{1}, e^{l_q(-z)})w$ when $0 \leq \arg z < \pi$ and is equal to $\Omega_0(\mathcal{Y})(\mathbf{1}, e^{l_q(-z)})w$ when $\pi \leq \arg z < 2\pi$. By Proposition 3.44, $\Omega_{-1}(\mathcal{Y})$ and $\Omega_0(\mathcal{Y})$ are logarithmic intertwining operators of type $\binom{W_3}{VW}$. As in Example 4.23, we see that $\Omega_{-1}(\mathcal{Y})(\mathbf{1}, y)$ and $\Omega_0(\mathcal{Y})(\mathbf{1}, y)$ are equal to $\mathbf{1}_{-1,0}^{\Omega_{-1}(\mathcal{Y})}$ and $\mathbf{1}_{-1,0}^{\Omega_0(\mathcal{Y})}$, respectively, and these maps preserve (generalized) weights. Therefore η is a linear map from W to W_3 preserving (generalized) weights. Using the Jacobi identity for the $P(z)$ -intertwining map I and the

fact that $Y(u, x_1)\mathbf{1} \in V[[x_1]]$, we have

$$\begin{aligned}
\eta(Y_W(u, x_0)w) &= e^{-zL(-1)}I(Y_W(u, x_0)w \otimes \mathbf{1}) \\
&= \text{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) e^{-zL(-1)}Y_3(u, x_1)I(w \otimes \mathbf{1}) \\
&\quad - \text{Res}_{x_1} x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) e^{-zL(-1)}I(w \otimes Y(u, x_1)\mathbf{1}) \\
&= e^{-zL(-1)}Y_3(u, x_0 + z)I(w \otimes \mathbf{1}) \\
&= Y_3(u, x_0)e^{-zL(-1)}I(w \otimes \mathbf{1}) \\
&= Y_3(u, x_0)\eta(w)
\end{aligned}$$

for $u \in V$ and $w \in W$, proving that η is a module map. For $w \in W$,

$$\begin{aligned}
(\bar{\eta} \circ (\Omega_p(Y_W)(\cdot, z)\cdot))(w \otimes \mathbf{1}) &= \bar{\eta}(e^{zL(-1)}Y_W(\mathbf{1}, -z)w) \\
&= e^{zL(-1)}\eta(w) \\
&= e^{zL(-1)}e^{-zL(-1)}I(w \otimes \mathbf{1}) \\
&= I(w \otimes \mathbf{1}).
\end{aligned} \tag{4.46}$$

Since both $\bar{\eta} \circ (\Omega_p(Y_W)(\cdot, z)\cdot)$ and I are $P(z)$ -intertwining maps of type $\binom{W_3}{WV}$, using the Jacobi identity for $P(z)$ -intertwining operators and (4.46) (cf. Example 4.23), we have

$$(\bar{\eta} \circ (\Omega_p(Y_W)(\cdot, z)\cdot))(w \otimes v) = I(w \otimes v)$$

for $v \in V$ and $w \in W$, proving (4.44). Thus $(W, Y_W; \Omega_p(Y_W)(\cdot, z)\cdot)$ is the $P(z)$ -tensor product of W and V in \mathcal{C} .

We discussed the important special class of finitely reductive vertex operator algebras in the Introduction. In case V is a finitely reductive vertex operator algebra, the $P(z)$ -tensor product always exists, as we are about to establish (following [HL5] and [HL7]). As in the Introduction, the definition of finite reductivity is:

Definition 4.25 A vertex operator algebra V is *finitely reductive* if

1. Every V -module is completely reducible.
2. There are only finitely many irreducible V -modules (up to equivalence).
3. All the fusion rules (the dimensions of the spaces of intertwining operators among triples of modules) for V are finite.

Remark 4.26 In this case, every V -module is of course a *finite* direct sum of irreducible modules. Also, the third condition holds if the finiteness of the fusion rules among triples of only *irreducible* modules is assumed.

Remark 4.27 We are of course taking the notion of V -module so that the grading restriction conditions are the ones described in Remark 2.27, formulas (2.89) and (2.90); in particular, V -modules are understood to be \mathbb{C} -graded. Recall from Remark 2.20 that for an irreducible module, all its weights are congruent to one another modulo \mathbb{Z} . Thus for an irreducible module, our grading-truncation condition (2.89) amounts exactly to the condition that the real parts of the weights are bounded from below. In [HL5]–[HL7], boundedness of the real parts of the weights from below was our grading-truncation condition in the definition of the notion of module for a vertex operator algebra. Thus the first two conditions in the notion of finite reductivity are the same whether we use the current grading restriction conditions in the definition of the notion of module or the corresponding conditions in [HL5]–[HL7]. As for intertwining operators, recall from Remark 3.13 and Corollary 3.23 that when the first two conditions are satisfied, the notion of (ordinary, non-logarithmic) intertwining operator here coincides with that in [HL5] because the truncation conditions agree. Also, in this setting, by Remark 3.24, the logarithmic and ordinary intertwining operators are the same, and so the spaces of intertwining operators $\mathcal{V}_{W_1 W_2}^{W_3}$ and fusion rules $N_{W_1 W_2}^{W_3}$ in Definition 3.18 have the same meanings as in [HL5]. Thus the notion of finite reductivity for a vertex operator algebra is the same whether we use the current grading restriction and truncation conditions in the definitions of the notions of module and of intertwining operator or the corresponding conditions in [HL5]–[HL7]. In particular, finite reductivity of V according to Definition 4.25 is equivalent to the corresponding notion, “rationality” (recall the Introduction) in [HL5]–[HL7].

Remark 4.28 For a vertex operator algebra V (in particular, a finitely reductive one), the category \mathcal{M} of V -modules coincides with the category \mathcal{M}_{sg} of strongly graded V -modules; recall Notation 2.36.

For the rest of Section 4.1, let us assume that V is a finitely reductive vertex operator algebra. We shall now show that $P(z)$ -tensor products always exist in the category \mathcal{M} ($= \mathcal{M}_{sg}$) of V -modules, in the sense of Definition 4.13.

Consider V -modules W_1 , W_2 and W_3 . We know that

$$N_{W_1 W_2}^{W_3} = \dim \mathcal{V}_{W_1 W_2}^{W_3} < \infty \quad (4.47)$$

and from Proposition 4.7, we also have

$$N_{W_1 W_2}^{W_3} = \dim \mathcal{M}[P(z)]_{W_1 W_2}^{W_3} = \dim \mathcal{M}_{W_1 W_2}^{W_3} < \infty \quad (4.48)$$

(recall Definition 4.2).

The natural evaluation map

$$\begin{aligned} W_1 \otimes W_2 \otimes \mathcal{M}_{W_1 W_2}^{W_3} &\rightarrow \overline{W}_3 \\ w_{(1)} \otimes w_{(2)} \otimes I &\mapsto I(w_{(1)} \otimes w_{(2)}) \end{aligned} \quad (4.49)$$

gives a natural map

$$\mathcal{F}[P(z)]_{W_1 W_2}^{W_3} : W_1 \otimes W_2 \rightarrow \text{Hom}(\mathcal{M}_{W_1 W_2}^{W_3}, \overline{W}_3) = (\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes \overline{W}_3. \quad (4.50)$$

Since $\dim \mathcal{M}_{W_1 W_2}^{W_3} < \infty$, $(\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes W_3$ is a V -module (with finite-dimensional weight spaces) in the obvious way, and the map $\mathcal{F}[P(z)]_{W_1 W_2}^{W_3}$ is clearly a $P(z)$ -intertwining map, where we make the identification

$$(\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes \overline{W_3} = \overline{(\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes W_3}. \quad (4.51)$$

This gives us a natural $P(z)$ -product for the category $\mathcal{M} = \mathcal{M}_{sg}$ (recall Definition 4.11). Moreover, we have a natural linear injection

$$\begin{aligned} i : \mathcal{M}_{W_1 W_2}^{W_3} &\rightarrow \text{Hom}_V((\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes W_3, W_3) \\ I &\mapsto (f \otimes w_{(3)}) \mapsto f(I)w_{(3)} \end{aligned} \quad (4.52)$$

which is an isomorphism if W_3 is irreducible, since in this case,

$$\text{Hom}_V(W_3, W_3) \simeq \mathbb{C}$$

(see [FHL], Remark 4.7.1). On the other hand, the natural map

$$\begin{aligned} h : \text{Hom}_V((\mathcal{M}_{W_1 W_2}^{W_3})^* \otimes W_3, W_3) &\rightarrow \mathcal{M}_{W_1 W_2}^{W_3} \\ \eta &\mapsto \bar{\eta} \circ \mathcal{F}[P(z)]_{W_1 W_2}^{W_3} \end{aligned} \quad (4.53)$$

given by composition clearly satisfies the condition that

$$h(i(I)) = I, \quad (4.54)$$

so that if W_3 is irreducible, the maps h and i are mutually inverse isomorphisms and we have the universal property that for any $I \in \mathcal{M}_{W_1 W_2}^{W_3}$, there exists a unique η such that

$$I = \bar{\eta} \circ \mathcal{I}_{W_1 W_2}^{W_3} \quad (4.55)$$

(cf. Definition 4.13).

Using this, we can now show, in the next result, that $P(z)$ -tensor products always exist for the category of modules for a finitely reductive vertex operator algebra, and we shall in fact exhibit the $P(z)$ -tensor product. Note that there is no need to assume that W_1 and W_2 are irreducible in the formulation or proof, but by Proposition 4.21, the case in which W_1 and W_2 are irreducible is in fact sufficient, and the tensor product operation is canonically described using only the spaces of intertwining maps among triples of *irreducible* modules.

Proposition 4.29 *Let V be a finitely reductive vertex operator algebra and let W_1 and W_2 be V -modules. Then $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ exists, and in fact*

$$W_1 \boxtimes_{P(z)} W_2 = \prod_{i=1}^k (\mathcal{M}_{W_1 W_2}^{M_i})^* \otimes M_i, \quad (4.56)$$

where $\{M_1, \dots, M_k\}$ is a set of representatives of the equivalence classes of irreducible V -modules, and the right-hand side of (4.56) is equipped with the V -module and $P(z)$ -product structure indicated above. That is,

$$\boxtimes_{P(z)} = \sum_{i=1}^k \mathcal{F}[P(z)]_{W_1 W_2}^{M_i}. \quad (4.57)$$

Proof From the comments above and the definitions, it is clear that we have a $P(z)$ -product. Let $(W_3, Y_3; I)$ be any $P(z)$ -product. Then $W_3 = \coprod_j U_j$ where j ranges through a finite set and each U_j is irreducible. Let $\pi_j : W_3 \rightarrow U_j$ denote the j -th projection. A module map $\eta : \coprod_{i=1}^k (\mathcal{M}_{W_1 W_2}^{M_i})^* \otimes M_i \rightarrow W_3$ amounts to module maps

$$\eta_{ij} : (\mathcal{M}_{W_1 W_2}^{M_i})^* \otimes M_i \rightarrow U_j$$

for each i and j such that $U_j \simeq M_i$, and $I = \bar{\eta} \circ \boxtimes_{P(z)}$ if and only if

$$\bar{\pi}_j \circ I = \bar{\eta}_{ij} \circ \mathcal{F}_{W_1 W_2}^{M_i}$$

for each i and j , the bars having the obvious meaning. But $\bar{\pi}_j \circ I$ is a $P(z)$ -intertwining map of type $\binom{U_j}{W_1 W_2}$, and so $\bar{\iota} \circ \bar{\pi}_j \circ I \in \mathcal{M}_{W_1 W_2}^{M_i}$, where $\iota : U_j \xrightarrow{\sim} M_i$ is a fixed isomorphism. Denote this map by τ . Thus what we finally want is a unique module map

$$\theta : (\mathcal{M}_{W_1 W_2}^{M_i})^* \otimes M_i \rightarrow M_i$$

such that

$$\tau = \bar{\theta} \circ \mathcal{F}[P(z)]_{W_1 W_2}^{M_i}.$$

But we in fact have such a unique θ , by (4.54)–(4.55). \square

Remark 4.30 By combining Proposition 4.29 with Proposition 4.7, we can express $W_1 \boxtimes_{P(z)} W_2$ in terms of $\mathcal{V}_{W_1 W_2}^{M_i}$ in place of $\mathcal{M}_{W_1 W_2}^{M_i}$.

Remark 4.31 If we know the fusion rules among triples of irreducible V -modules, then from Proposition 4.29 we know all the $P(z)$ -tensor product modules, up to equivalence; that is, we know the multiplicity of each irreducible V -module in each $P(z)$ -tensor product module. But recall that the $P(z)$ -tensor product structure of $W_1 \boxtimes_{P(z)} W_2$ involves much more than just the V -module structure.

As we discussed in the Introduction, the main theme of this work is to construct natural “associativity” isomorphisms between triple tensor products of the shape $W_1 \boxtimes (W_2 \boxtimes W_3)$ and $(W_1 \boxtimes W_2) \boxtimes W_3$, for (generalized) modules W_1, W_2 and W_3 . In the finitely reductive case, let W_1, W_2 and W_3 be V -modules. By Proposition 4.29, we have, as V -modules,

$$\begin{aligned} W_1 \boxtimes_{P(z)} (W_2 \boxtimes_{P(z)} W_3) &= W_1 \boxtimes_{P(z)} \left(\prod_{i=1}^k M_i \otimes (\mathcal{M}_{W_2 W_3}^{M_i})^* \right) \\ &= \prod_{i=1}^k (W_1 \boxtimes_{P(z)} M_i) \otimes (\mathcal{M}_{W_2 W_3}^{M_i})^* \\ &= \prod_{i=1}^k \left(\prod_{j=1}^k (\mathcal{M}_{W_1 M_i}^{M_j})^* \otimes M_j \right) \otimes (\mathcal{M}_{W_2 W_3}^{M_i})^* \end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^k \left(\prod_{i=1}^k (\mathcal{M}_{W_1 M_i}^{M_j})^* \otimes (\mathcal{M}_{W_2 W_3}^{M_i})^* \right) \otimes M_j \\
&= \prod_{j=1}^k \left(\prod_{i=1}^k (\mathcal{M}_{W_1 M_i}^{M_j} \otimes \mathcal{M}_{W_2 W_3}^{M_i})^* \right) \otimes M_j
\end{aligned} \tag{4.58}$$

and

$$\begin{aligned}
(W_1 \boxtimes_{P(z)} W_2) \boxtimes_{P(z)} W_3 &= \left(\prod_{i=1}^k M_i \otimes (\mathcal{M}_{W_1 W_2}^{M_i})^* \right) \boxtimes_{P(z)} W_3 \\
&= \prod_{i=1}^k (M_i \boxtimes_{P(z)} W_3) \otimes (\mathcal{M}_{W_1 W_2}^{M_i})^* \\
&= \prod_{i=1}^k \left(\prod_{j=1}^k (\mathcal{M}_{M_i W_3}^{M_j})^* \otimes M_j \right) \otimes (\mathcal{M}_{W_1 W_2}^{M_i})^* \\
&= \prod_{j=1}^k \left(\prod_{i=1}^k (\mathcal{M}_{M_i W_3}^{M_j})^* \otimes (\mathcal{M}_{W_1 W_2}^{M_i})^* \right) \otimes M_j \\
&= \prod_{j=1}^k \left(\prod_{i=1}^k (\mathcal{M}_{M_i W_3}^{M_j} \otimes \mathcal{M}_{W_1 W_2}^{M_i})^* \right) \otimes M_j.
\end{aligned} \tag{4.59}$$

These two V -modules will be equivalent if for each $j = 1, \dots, k$, their M_j -multiplicities are the same, that is, if

$$\sum_{i=1}^k N_{W_1 M_i}^{M_j} N_{W_2 W_3}^{M_i} = \sum_{i=1}^k N_{W_1 W_2}^{M_i} N_{M_i W_3}^{M_j}. \tag{4.60}$$

However, knowing only that these two V -modules are equivalent (knowing that \boxtimes is “associative” in only a rough sense) is far from enough. What we need is a natural isomorphism between these two modules analogous to the natural isomorphism

$$\mathcal{W}_1 \otimes (\mathcal{W}_2 \otimes \mathcal{W}_3) \xrightarrow{\sim} (\mathcal{W}_1 \otimes \mathcal{W}_2) \otimes \mathcal{W}_3 \tag{4.61}$$

of vector spaces \mathcal{W}_i determined by the natural condition

$$w_{(1)} \otimes (w_{(2)} \otimes w_{(3)}) \mapsto (w_{(1)} \otimes w_{(2)}) \otimes w_{(3)} \tag{4.62}$$

on elements (recall the Introduction). Suppose that \mathcal{W}_1 , \mathcal{W}_2 and \mathcal{W}_3 are finite-dimensional completely reducible modules for some Lie algebra. Then we of course have the analogue of the relation (4.60). But knowing the equality of these multiplicities certainly does not give the natural isomorphism (4.61)–(4.62).

Our intent to construct a natural isomorphism between the spaces (4.58) and (4.59) (under suitable conditions) in fact provides a guide to what we need to do. In (4.58), each

space $\mathcal{M}_{W_1 M_i}^{M_j} \otimes \mathcal{M}_{W_2 W_3}^{M_i}$ suggests combining an intertwining map \mathcal{Y}_1 of type $\binom{M_j}{W_1 M_i}$ with an intertwining map \mathcal{Y}_2 of type $\binom{M_i}{W_2 W_3}$, presumably by composition:

$$\mathcal{Y}_1(w_{(1)}, z) \mathcal{Y}_2(w_{(2)}, z). \quad (4.63)$$

But this will not work, since this composition does not exist because the relevant formal series in z does not converge; we must instead take

$$\mathcal{Y}_1(w_{(1)}, z_1) \mathcal{Y}_2(w_{(2)}, z_2), \quad (4.64)$$

where the complex numbers z_1 and z_2 are such that

$$|z_1| > |z_2| > 0,$$

by analogy with, and generalizing, the situation in Corollary 2.42. The composition (4.64) must be understood using convergence and “matrix coefficients,” again as in Corollary 2.42.

Similarly, in (4.59), each space $\mathcal{M}_{M_i W_3}^{M_j} \otimes \mathcal{M}_{W_1 W_2}^{M_i}$ suggests combining an intertwining map \mathcal{Y}^1 of type $\binom{M_j}{M_i W_3}$ with an intertwining map of type \mathcal{Y}^2 of type $\binom{M_i}{W_1 W_2}$:

$$\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_1 - z_2)w_{(2)}, z_2),$$

a (convergent) iterate of intertwining maps as in (2.114), with

$$|z_2| > |z_1 - z_2| > 0,$$

not

$$\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z), z), \quad (4.65)$$

which fails to converge.

The natural way to construct a natural associativity isomorphism between (4.58) and (4.59) will in fact, then, be to implement a correspondence of the type

$$\mathcal{Y}_1(w_{(1)}, z_1) \mathcal{Y}_2(w_{(2)}, z_2) = \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_1 - z_2)w_{(2)}, z_2), \quad (4.66)$$

as we have previewed in the Introduction (formula (1.34)) and also in (2.114). Formula (4.66) expresses the existence and associativity of the general nonmeromorphic operator product expansion, as discussed in Remark 2.44. Note that this viewpoint shows that we should not try directly to construct a natural isomorphism

$$W_1 \boxtimes_{P(z)} (W_2 \boxtimes_{P(z)} W_3) \xrightarrow{\sim} (W_1 \boxtimes_{P(z)} W_2) \boxtimes_{P(z)} W_3, \quad (4.67)$$

but rather a natural isomorphism

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \xrightarrow{\sim} (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3. \quad (4.68)$$

This is what we will actually do in this work, in the general logarithmic, not-necessarily-finitely-reductive case, under suitable conditions. The natural isomorphism (4.68) will act as follows on elements of the completions of the relevant (generalized) modules:

$$w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \mapsto (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}, \quad (4.69)$$

implementing the strategy suggested by the classical natural isomorphism (4.61)–(4.62). Recall that we previewed this strategy in the Introduction.

It turns out that in order to carry out this program, including the construction of equalities of the type (4.66) (the existence and associativity of the nonmeromorphic operator product expansion) in general, we cannot use the realization of the $P(z)$ -tensor product given in Proposition 4.29, *even when V is a finitely reductive vertex operator algebra*. As in [HL5]–[HL7] and [H1], what we do instead is to construct $P(z)$ -tensor products in a completely different way (even in the finitely reductive case), a way that allows us to also construct the natural associativity isomorphisms. Section 5 is devoted to this construction of $P(z)$ - (and $Q(z)$ -) tensor products.

4.2 The notion of $Q(z)$ -tensor product

We now generalize the notion of $Q(z)$ -tensor product of modules from [HL5] to the setting of the present work, parallel to what we did for the $P(z)$ -tensor product above. Here we give only the results that we will need later. Other results similar to those for $P(z)$ -tensor products certainly also carry over to the case of $Q(z)$, for example, the results above on the finitely reductive case, as were presented in [HL5].

Definition 4.32 Let (W_1, Y_1) , (W_2, Y_2) and (W_3, Y_3) be generalized V -modules. A $Q(z)$ -intertwining map of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ is a linear map

$$I : W_1 \otimes W_2 \rightarrow \overline{W}_3$$

such that the following conditions are satisfied: the *grading compatibility condition*: for $\beta, \gamma \in \tilde{A}$ and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$,

$$I(w_{(1)} \otimes w_{(2)}) \in \overline{W}_3^{(\beta+\gamma)}; \quad (4.70)$$

the *lower truncation condition*: for any elements $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and any $n \in \mathbb{C}$,

$$\pi_{n-m} I(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large} \quad (4.71)$$

(which follows from (4.70), in view of the grading restriction condition (2.85); cf. (4.3)); the *Jacobi identity*:

$$\begin{aligned} & z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_3^o(v, x_0) I(w_{(1)} \otimes w_{(2)}) \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) I(Y_1^o(v, x_1) w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) I(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \end{aligned} \quad (4.72)$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (recall (2.57) for the notation Y° , and note that the left-hand side of (4.72) is meaningful because any infinite linear combination of v_n of the form $\sum_{n < N} a_n v_n$ ($a_n \in \mathbb{C}$) acts on any $I(w_{(1)} \otimes w_{(2)})$, in view of (4.71)); and the $\mathfrak{sl}(2)$ -*bracket relations*: for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\begin{aligned} L(-j)I(w_{(1)} \otimes w_{(2)}) &= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i I((L(-j+i)w_{(1)}) \otimes w_{(2)}) \\ &\quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i I(w_{(1)} \otimes L(j-i)w_{(2)}) \end{aligned} \quad (4.73)$$

for $j = -1, 0$ and 1 (note that if V is in fact a conformal vertex algebra, this follows automatically from (4.72) by setting $v = \omega$ and taking $\text{Res}_{x_1} \text{Res}_{x_0} x_0^{j+1}$). The vector space of $Q(z)$ -intertwining maps of type $\binom{W_3}{W_1 W_2}$ is denoted by $\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}$.

Remark 4.33 As was explained in [HL5], the symbol $Q(z)$ represents the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with one negatively oriented puncture at z and two ordered positively oriented punctures at ∞ and 0 , with local coordinates $w - z$, $1/w$ and w , respectively, vanishing at these punctures. In fact, this structure is conformally equivalent to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with one negatively oriented puncture at ∞ and two ordered positively oriented punctures $1/z$ and 0 , with local coordinates $z/(zw - 1)$, $(zw - 1)/z^2 w$ and $z^2 w/(zw - 1)$ vanishing at ∞ , $1/z$ and 0 , respectively.

Remark 4.34 In the case of \mathbb{C} -graded ordinary modules for a vertex operator algebra, where the grading restriction condition (2.89) for a module W is replaced by the (more restrictive) condition

$$W_{(n)} = 0 \quad \text{for } n \in \mathbb{C} \text{ with sufficiently negative real part} \quad (4.74)$$

as in [HL5] (and where, in our context, the abelian groups A and \tilde{A} are trivial), the notion of $Q(z)$ -intertwining map above agrees with the earlier one introduced in [HL5]; in this case, the conditions (4.70) and (4.71) are automatic.

In view of Remarks 4.3 and 4.33, we can now give a natural correspondence between $P(z)$ - and $Q(z)$ -intertwining maps. (See the next three results.) Recall that since our generalized V -modules are strongly graded, we have contragredient generalized modules of generalized modules.

Proposition 4.35 *Let $I : W_1 \otimes W_2 \rightarrow \overline{W_3}$ and $J : W'_3 \otimes W_2 \rightarrow \overline{W'_1}$ be linear maps related to each other by:*

$$\langle w_{(1)}, J(w'_{(3)} \otimes w_{(2)}) \rangle = \langle w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle \quad (4.75)$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$. Then I is a $Q(z)$ -intertwining map of type $\binom{W_3}{W_1 W_2}$ if and only if J is a $P(z)$ -intertwining map of type $\binom{W'_1}{W'_3 W_2}$.

Proof Suppose that I is a $Q(z)$ -intertwining map of type $(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix})$. We shall show that J is a $P(z)$ -intertwining map of type $(\begin{smallmatrix} W'_1 \\ W'_3 W'_2 \end{smallmatrix})$.

Since I satisfies the grading compatibility condition, it is clear that J also satisfies this condition. For the lower truncation condition for J , it suffices to show that for any $w_{(2)} \in W_2^{(\beta)}$ and $w'_{(3)} \in (W'_3)^{(\gamma)}$, where $\beta, \gamma \in \tilde{A}$, and any $n \in \mathbb{C}$, $\langle \pi_{[n-m]} W_1^{(-\beta-\gamma)}, J(w'_{(3)} \otimes w_{(2)}) \rangle = 0$ for $m \in \mathbb{N}$ sufficiently large, or that

$$\langle w'_{(3)}, I(\pi_{[n-m]} W_1^{(-\beta-\gamma)} \otimes w_{(2)}) \rangle = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large.} \quad (4.76)$$

But (4.76) follows immediately from (2.85).

Now we prove the Jacobi identity for J . The Jacobi identity for I gives

$$\begin{aligned} & z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \langle w'_{(3)}, Y_3^o(v, x_0) I(w_{(1)} \otimes w_{(2)}) \rangle \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \langle w'_{(3)}, I(Y_1^o(v, x_1) w_{(1)} \otimes w_{(2)}) \rangle \\ &\quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \langle w'_{(3)}, I(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \rangle \end{aligned} \quad (4.77)$$

for any $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$. By (2.73) the left-hand side is equal to

$$z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \langle Y'_3(v, x_0) w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle$$

So by (4.75), the identity (4.77) can be written as

$$\begin{aligned} & z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \langle w_{(1)}, J(Y'_3(v, x_0) w'_{(3)} \otimes w_{(2)}) \rangle \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \langle Y_1^o(v, x_1) w_{(1)}, J(w'_{(3)} \otimes w_{(2)}) \rangle \\ &\quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \langle w_{(1)}, J(w'_{(3)} \otimes Y_2(v, x_1) w_{(2)}) \rangle. \end{aligned}$$

Applying (2.73) to the first term of the right-hand side we see that this can be written as

$$\begin{aligned} & z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \langle w_{(1)}, J(Y'_3(v, x_0) w'_{(3)} \otimes w_{(2)}) \rangle \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \langle w_{(1)}, Y'_1(v, x_1) J(w'_{(3)} \otimes w_{(2)}) \rangle \\ &\quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \langle w_{(1)}, J(w'_{(3)} \otimes Y_2(v, x_1) w_{(2)}) \rangle \end{aligned}$$

for any $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$. This is exactly the Jacobi identity for J .

The $\mathfrak{sl}(2)$ -bracket relations can be proved similarly, as follows: The $\mathfrak{sl}(2)$ -bracket relations for I give

$$\begin{aligned}\langle w'_{(3)}, L(-j)I(w_{(1)} \otimes w_{(2)}) \rangle &= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \langle w'_{(3)}, I((L(-j+i)w_{(1)}) \otimes w_{(2)}) \rangle \\ &\quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \langle w'_{(3)}, I(w_{(1)} \otimes L(j-i)w_{(2)}) \rangle\end{aligned}$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$ and $j = -1, 0, 1$. Using (2.75) and then applying (4.75) we get

$$\begin{aligned}\langle w_{(1)}, J(L'(j)w'_{(3)} \otimes w_{(2)}) \rangle &= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \langle L(-j+i)w_{(1)}, J(w'_{(3)} \otimes w_{(2)}) \rangle \\ &\quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \langle w_{(1)}, J(w'_{(3)} \otimes L(j-i)w_{(2)}) \rangle,\end{aligned}$$

or

$$\begin{aligned}J(L'(j)w'_{(3)} \otimes w_{(2)}) &= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i L(j-i)J(w'_{(3)} \otimes w_{(2)}) \\ &\quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i J(w'_{(3)} \otimes L(j-i)w_{(2)}),\end{aligned}$$

for $j = -1, 0, 1$. This is the alternative form (4.7) of the $\mathfrak{sl}(2)$ -bracket relations for J . Hence J is a $P(z)$ -intertwining map.

The other direction of the proposition is proved by simply reversing the order of the arguments. \square

Let W_1 , W_2 and W_3 be generalized V -modules, as above. We shall call an element λ of $(W_1 \otimes W_2 \otimes W_3)^*$ \tilde{A} -compatible if

$$\lambda((W_1)^{(\beta)} \otimes (W_2)^{(\gamma)} \otimes (W_3)^{(\delta)}) = 0$$

for $\beta, \gamma, \delta \in \tilde{A}$ satisfying $\beta + \gamma + \delta \neq 0$. Recall from Definitions 2.18 and 2.32 that for a generalized V -module W , $\overline{W'}$ can be viewed as a (usually proper) subspace of W^* . We shall call a linear map

$$I : W_1 \otimes W_2 \rightarrow W_3^*$$

\tilde{A} -compatible if its image lies in $\overline{W'_3}$, that is,

$$I : W_1 \otimes W_2 \rightarrow \overline{W'_3},$$

and if I satisfies the usual grading compatibility condition (4.2) or (4.70) for $P(z)$ - or $Q(z)$ -intertwining maps. Now an element λ of $(W_1 \otimes W_2 \otimes W_3)^*$ amounts exactly to a linear map

$$I_\lambda : W_1 \otimes W_2 \rightarrow W_3^*.$$

If λ is \tilde{A} -compatible, then for $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$ and $w_{(3)} \in W_3^{(\delta)}$ such that $\delta \neq -(\beta + \gamma)$,

$$\langle w_{(3)}, I_\lambda(w_{(1)} \otimes w_{(2)}) \rangle = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0,$$

so that $I_\lambda(w_{(1)} \otimes w_{(2)}) \in \overline{(W_3')^{(\beta+\gamma)}}$ and I_λ is \tilde{A} -compatible. Similarly, if I_λ is \tilde{A} -compatible, then so is λ . Thus we have the following straightforward result relating \tilde{A} -compatibility of λ with that of I_λ :

Lemma 4.36 *The linear functional $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ is \tilde{A} -compatible if and only if I_λ is \tilde{A} -compatible. The map given by $\lambda \mapsto I_\lambda$ is the unique linear isomorphism from the space of \tilde{A} -compatible elements of $(W_1 \otimes W_2 \otimes W_3)^*$ to the space of \tilde{A} -compatible linear maps from $W_1 \otimes W_2$ to $\overline{W_3'}$ such that*

$$\langle w_{(3)}, I_\lambda(w_{(1)} \otimes w_{(2)}) \rangle = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Similarly, there are canonical linear isomorphisms from the space of \tilde{A} -compatible elements of $(W_1 \otimes W_2 \otimes W_3)^*$ to the space of \tilde{A} -compatible linear maps from $W_1 \otimes W_3$ to $\overline{W_2'}$ and to the space of \tilde{A} -compatible linear maps from $W_2 \otimes W_3$ to $\overline{W_1'}$ satisfying the corresponding conditions. In particular, there is a canonical linear isomorphism from the space of \tilde{A} -compatible linear maps from $W_1 \otimes W_2$ to $\overline{W_3'}$ to the space of \tilde{A} -compatible linear maps from $W_3' \otimes W_2$ to $\overline{W_1'}$ given by (4.75). \square

Using this lemma and Proposition 4.35, we have:

Corollary 4.37 *The formula (4.75) gives a canonical linear isomorphism between the space of $Q(z)$ -intertwining maps of type $\binom{W_3}{W_1 W_2}$ and the space of $P(z)$ -intertwining maps of type $\binom{W_1'}{W_3' W_2}$. \square*

We can now use Proposition 4.7 together with Proposition 4.35 and Corollary 4.37 to construct a correspondence between the logarithmic intertwining operators of type $\binom{W_1'}{W_3' W_2}$ and the $Q(z)$ -intertwining maps of type $\binom{W_3}{W_1 W_2}$; this generalizes the corresponding result in the finitely reductive case, with ordinary modules, in [HL5]. Fix an integer p . Let \mathcal{Y} be a logarithmic intertwining operator of type $\binom{W_1'}{W_3' W_2}$, and use (4.14) to define a linear map $I_{\mathcal{Y},p} : W_3' \otimes W_2 \rightarrow \overline{W_1'}$; by Proposition 4.7, this is a $P(z)$ -intertwining map of the same type. Then use Proposition 4.35 and Corollary 4.37 to define a $Q(z)$ -intertwining map $I_{\mathcal{Y},p}^{Q(z)} : W_1 \otimes W_2 \rightarrow \overline{W_3}$ of type $\binom{W_3}{W_1 W_2}$ (uniquely) by

$$\begin{aligned} \langle w'_{(3)}, I_{\mathcal{Y},p}^{Q(z)}(w_{(1)} \otimes w_{(2)}) \rangle_{W_3} &= \langle w_{(1)}, I_{\mathcal{Y},p}(w'_{(3)} \otimes w_{(2)}) \rangle_{W_1'} \\ &= \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)} w_{(2)}) \rangle_{W_1'} \end{aligned} \tag{4.78}$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$. (We are using the symbol $Q(z)$ to distinguish this from the $P(z)$ case above.) Then the correspondence $\mathcal{Y} \mapsto I_{\mathcal{Y},p}^{Q(z)}$ is an isomorphism from $\mathcal{V}_{W'_3 W_2}^{W'_1}$ to $\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}$. From Proposition 4.7 and (4.17), its inverse is given by sending a $Q(z)$ -intertwining map I of type $\binom{W_3}{W_1 W_2}$ to the logarithmic intertwining operator $\mathcal{Y}_{I,p}^{Q(z)} : W'_3 \otimes W_2 \rightarrow W'_1[\log x]\{x\}$ defined by

$$\begin{aligned} & \langle w_{(1)}, \mathcal{Y}_{I,p}^{Q(z)}(w'_{(3)}, x)w_{(2)} \rangle_{W'_1} \\ &= \langle y^{-L'(0)} x^{-L'(0)} w'_{(3)}, I(y^{L(0)} x^{L(0)} w_{(1)} \otimes y^{-L(0)} x^{-L(0)} w_{(2)}) \rangle_{W_3} \Big|_{y=e^{-L_p(z)}} \end{aligned}$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$. Thus we have:

Proposition 4.38 *For $p \in \mathbb{Z}$, the correspondence $\mathcal{Y} \mapsto I_{\mathcal{Y},p}^{Q(z)}$ is a linear isomorphism from the space $\mathcal{V}_{W'_3 W_2}^{W'_1}$ of logarithmic intertwining operators of type $\binom{W'_1}{W'_3 W_2}$ to the space $\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}$ of $Q(z)$ -intertwining maps of type $\binom{W_3}{W_1 W_2}$. Its inverse is given by $I \mapsto \mathcal{Y}_{I,p}^{Q(z)}$. \square*

We now give the definition of $Q(z)$ -tensor product.

Definition 4.39 Let \mathcal{C}_1 be either \mathcal{M}_{sg} or \mathcal{GM}_{sg} . For $W_1, W_2 \in \text{ob } \mathcal{C}_1$, a $Q(z)$ -product of W_1 and W_2 is an object (W_3, Y_3) of \mathcal{C}_1 together with a $Q(z)$ -intertwining map I_3 of type $\binom{W_3}{W_1 W_2}$. We denote it by $(W_3, Y_3; I_3)$ or simply by (W_3, I_3) . Let $(W_4, Y_4; I_4)$ be another $Q(z)$ -product of W_1 and W_2 . A *morphism* from $(W_3, Y_3; I_3)$ to $(W_4, Y_4; I_4)$ is a module map η from W_3 to W_4 such that the diagram

$$\begin{array}{ccc} & W_1 \otimes W_2 & \\ I_3 \swarrow & & \searrow I_4 \\ \overline{W}_3 & \xrightarrow{\quad \bar{\eta} \quad} & \overline{W}_4 \end{array}$$

commutes, that is,

$$I_4 = \bar{\eta} \circ I_3.$$

where, as before, $\bar{\eta}$ is the natural map from \overline{W}_3 to \overline{W}_4 uniquely extending η .

Definition 4.40 Let \mathcal{C} be a full subcategory of either \mathcal{M}_{sg} or \mathcal{GM}_{sg} . For $W_1, W_2 \in \text{ob } \mathcal{C}$, a $Q(z)$ -tensor product of W_1 and W_2 in \mathcal{C} is a $Q(z)$ -product $(W_0, Y_0; I_0)$ with $W_0 \in \text{ob } \mathcal{C}$ such that for any $Q(z)$ -product $(W, Y; I)$ with $W \in \text{ob } \mathcal{C}$, there is a unique morphism from $(W_0, Y_0; I_0)$ to $(W, Y; I)$. Clearly, a $Q(z)$ -tensor product of W_1 and W_2 in \mathcal{C} , if it exists, is unique up to unique isomorphism. In this case we will denote it as $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ and call the object $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)})$ the $Q(z)$ -tensor product module of W_1 and W_2 in \mathcal{C} . Again we will skip the phrase “in \mathcal{C} ” if the category \mathcal{C} under consideration is clear in context.

The following immediate consequence of Definition 4.40 and Proposition 4.38 relates module maps from a $Q(z)$ -tensor product module with $Q(z)$ -intertwining maps and logarithmic intertwining operators:

Proposition 4.41 *Suppose that $W_1 \boxtimes_{Q(z)} W_2$ exists. We have a natural isomorphism*

$$\begin{aligned} \text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3} \\ \eta &\mapsto \bar{\eta} \circ \boxtimes_{Q(z)} \end{aligned}$$

and for $p \in \mathbb{Z}$, a natural isomorphism

$$\begin{aligned} \text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{V}_{W_3' W_2}^{W_1'} \\ \eta &\mapsto \mathcal{Y}_{\eta, p}^{Q(z)} \end{aligned}$$

where $\mathcal{Y}_{\eta, p}^{Q(z)} = \mathcal{Y}_{I, p}^{Q(z)}$ with $I = \bar{\eta} \circ \boxtimes_{Q(z)}$. \square

Suppose that the $Q(z)$ -tensor product $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ of W_1 and W_2 exists. We will sometimes denote the action of the canonical $Q(z)$ -intertwining map

$$w_{(1)} \otimes w_{(2)} \mapsto \boxtimes_{Q(z)}(w_{(1)} \otimes w_{(2)}) = \boxtimes_{Q(z)}(w_{(1)}, z)w_{(2)} \in \overline{W_1 \boxtimes_{Q(z)} W_2} \quad (4.79)$$

on elements simply by $w_{(1)} \boxtimes_{Q(z)} w_{(2)}$:

$$w_{(1)} \boxtimes_{Q(z)} w_{(2)} = \boxtimes_{Q(z)}(w_{(1)} \otimes w_{(2)}) = \boxtimes_{Q(z)}(w_{(1)}, z)w_{(2)}. \quad (4.80)$$

Using Propositions 3.44 and 3.46, we have the following result, generalizing Proposition 4.9 and Corollary 4.10 in [HL5]:

Proposition 4.42 *For any integer r , there is a natural isomorphism*

$$B_r : \mathcal{V}_{W_1 W_2}^{W_3} \rightarrow \mathcal{V}_{W_3' W_2}^{W_1'}$$

defined by the condition that for any logarithmic intertwining operator \mathcal{Y} in $\mathcal{V}_{W_1 W_2}^{W_3}$ and $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$,

$$\begin{aligned} &\langle w_{(1)}, B_r(\mathcal{Y})(w'_{(3)}, x)w_{(2)} \rangle_{W_1'} \\ &= \langle e^{-x^{-1}L(1)}w'_{(3)}, \mathcal{Y}(e^{xL(1)}w_{(1)}, x^{-1})e^{-xL(1)}e^{(2r+1)\pi i L(0)}(x^{-L(0)})^2 w_{(2)} \rangle_{W_3}. \end{aligned} \quad (4.81)$$

Proof From Proposition 3.44, for any integer r_1 we have an isomorphism Ω_{r_1} from $\mathcal{V}_{W_1 W_2}^{W_3}$ to $\mathcal{V}_{W_2 W_1}^{W_3}$, and from Proposition 3.46, for any integer r_2 we have an isomorphism A_{r_2} from $\mathcal{V}_{W_2 W_1}^{W_3}$ to $\mathcal{V}_{W_2 W_3'}^{W_1'}$. By Proposition 3.44 again, for any integer r_3 there is an isomorphism, which we again denote Ω_{r_3} , from $\mathcal{V}_{W_2 W_3'}^{W_1'}$ to $\mathcal{V}_{W_3' W_2}^{W_1'}$. Thus for any triple (r_1, r_2, r_3) of integers, we have an isomorphism $\Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_1}$ from $\mathcal{V}_{W_1 W_2}^{W_3}$ to $\mathcal{V}_{W_3' W_2}^{W_1'}$. Let \mathcal{Y} be a logarithmic

intertwining operator in $\mathcal{V}_{W_1 W_2}^{W_3}$ and $w_{(1)}, w_{(2)}, w'_{(3)}$ elements of W_1, W_2, W'_3 , respectively. From the definitions of Ω_{r_1}, A_{r_2} and Ω_{r_3} , we have

$$\begin{aligned}
& \langle (\Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_1})(\mathcal{Y})(w'_{(3)}, x)w_{(2)}, w_{(1)} \rangle_{W_1} = \\
& = \langle e^{xL(-1)} A_{r_2}(\Omega_{r_1}(\mathcal{Y}))(w_{(2)}, e^{(2r_3+1)\pi i} x)w'_{(3)}, w_{(1)} \rangle_{W_1} \\
& = \langle A_{r_2}(\Omega_{r_1}(\mathcal{Y}))(w_{(2)}, e^{(2r_3+1)\pi i} x)w'_{(3)}, e^{xL(1)} w_{(1)} \rangle_{W_1} \\
& = \langle w'_{(3)}, \Omega_{r_1}(\mathcal{Y})(e^{-xL(1)} e^{(2r_2+1)\pi i L(0)} e^{-2(2r_3+1)\pi i L(0)} (x^{-L(0)})^2 w_{(2)}, \\
& \quad e^{-(2r_3+1)\pi i} x^{-1}) e^{xL(1)} w_{(1)} \rangle_{W_3} \\
& = \langle w'_{(3)}, e^{-x^{-1}L(-1)} \mathcal{Y}(e^{xL(1)} w_{(1)}, e^{(2r_1+1)\pi i} e^{-(2r_3+1)\pi i} x^{-1}) \cdot \\
& \quad \cdot e^{-xL(1)} e^{(2r_2+1)\pi i L(0)} e^{-2(2r_3+1)\pi i L(0)} (x^{-L(0)})^2 w_{(2)} \rangle_{W_3} \\
& = \langle e^{-x^{-1}L(1)} w'_{(3)}, \mathcal{Y}(e^{xL(1)} w_{(1)}, e^{2(r_1-r_3)\pi i} x^{-1}) \cdot \\
& \quad \cdot e^{-xL(1)} e^{(2(r_2-2r_3-1)+1)\pi i L(0)} (x^{-L(0)})^2 w_{(2)} \rangle_{W_3}. \tag{4.82}
\end{aligned}$$

From (4.82) we see that $\Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_1}$ depends only on $r_2 - 2r_3 - 1$ and $r_1 - r_3$, and the operators $\Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_1}$ with different $r_1 - r_3$ but the same $r_2 - 2r_3 - 1$ differ from each other only by automorphisms of $\mathcal{V}_{W_1 W_2}^{W_3}$ (recall Remarks 3.30, 3.40 and 3.45). Thus for our purpose, we need only consider those isomorphisms such that $r_1 - r_3 = 0$. Given any integer r , we choose two integers r_2 and r_3 such that $r = r_2 - 2r_3 - 1$ and we define

$$B_r = \Omega_{r_3} \circ A_{r_2} \circ \Omega_{r_3}. \tag{4.83}$$

From (4.82) we see that B_r is independent of the choices of r_2 and r_3 and that (4.81) holds. \square

Combining the last two results, we obtain:

Corollary 4.43 *For any $W_1, W_2, W_3 \in \text{ob } \mathcal{C}$ such that $W_1 \boxtimes_{Q(z)} W_2$ exists and any integers p and r , we have a natural isomorphism*

$$\begin{aligned}
\text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) & \xrightarrow{\sim} \mathcal{V}_{W_1 W_2}^{W_3} \\
\eta & \mapsto B_r^{-1}(\mathcal{Y}_{\eta, p}^{Q(z)}). \quad \square \tag{4.84}
\end{aligned}$$

5 Constructions of $P(z)$ - and $Q(z)$ -tensor products

We now generalize the constructions of $P(z)$ - and $Q(z)$ -tensor products in [HL5]–[HL7] to the setting of the present work. In the earlier work [HL5]–[HL7] of the first two authors, the $Q(z)$ -tensor product of two modules was studied and developed first, in [HL5] and [HL6]. The $P(z)$ -tensor product was then studied systematically in [HL7], and many proofs for the $P(z)$ case were given by using the results established for the $Q(z)$ case in [HL5] and [HL6], rather than by carrying out the subtle arguments in the $P(z)$ case itself, arguments that are similar to (but different from) those in the $Q(z)$ case. In the present section and the next section, instead of following this approach of [HL5]–[HL7], we shall construct the $P(z)$ -tensor product and $Q(z)$ -tensor product of two modules independently. In particular, even for the finitely reductive case carried out in [HL5]–[HL7], some of the present results and proofs of the main theorems are completely new. One new result is Proposition 5.9 below, which was not stated or proved (or needed) in the finitely reductive case in [HL7]. This is proved below, by a direct argument in the $P(z)$ setting, rather than by the use of the $Q(z)$ structure. Theorems 5.39, 5.40, 5.70 and 5.71 formulated below will be proved in the next section. The proofs of Theorems 5.39 and 5.40 are new, even in the finitely reductive case. Recall Assumption 4.1.

5.1 Affinizations of vertex algebras and the opposite-operator map

Just as in [HL5]–[HL7], we shall use the Jacobi identity as a motivation to construct tensor products of (generalized) V -modules in a suitable category. To do this, we need to study various “affinizations” of a vertex algebra with respect to certain algebras and vector spaces of formal Laurent series and formal rational functions. The treatment of these matters below is very similar to that in [HL5], but here we must take into account the gradings by A and \tilde{A} . Here, as in Section 2 above, we are replacing the symbol $*$ for the “opposite-operator map” in [HL5] by o . In Subsections 5.2 and 5.3 below, we will be using the material in this subsection to construct certain actions $\tau_{P(z)}$ and $\tau_{Q(z)}$, in order to construct $P(z)$ - and $Q(z)$ -tensor products.

Let (W, Y_W) be a generalized V -module. We adjoin the formal variable t to our list of commuting formal variables. This variable will play a special role. Consider the vector spaces

$$V[t, t^{-1}] = V \otimes \mathbb{C}[t, t^{-1}] \subset V \otimes \mathbb{C}((t)) \subset V \otimes \mathbb{C}[[t, t^{-1}]] \subset V[[t, t^{-1}]]$$

(note carefully the distinction between the last two, since V is typically infinite-dimensional) and $W \otimes \mathbb{C}\{t\} \subset W\{t\}$ (recall (2.1)). The linear map

$$\begin{aligned} \tau_W : V[t, t^{-1}] &\rightarrow \text{End } W \\ v \otimes t^n &\mapsto v_n \end{aligned} \tag{5.1}$$

($v \in V, n \in \mathbb{Z}$) extends canonically to

$$\tau_W : V \otimes \mathbb{C}((t)) \rightarrow \text{End } W$$

$$v \otimes \sum_{n>N} a_n t^n \mapsto \sum_{n>N} a_n v_n \quad (5.2)$$

(but not to $V((t))$), in view of (2.49) and Assumption 4.1. It further extends canonically to

$$\tau_W : (V \otimes \mathbb{C}((t)))[[x, x^{-1}]] \rightarrow (\text{End } W)[[x, x^{-1}]], \quad (5.3)$$

where of course $(V \otimes \mathbb{C}((t)))[[x, x^{-1}]]$ can be viewed as the subspace of $V[[t, t^{-1}, x, x^{-1}]]$ such that the coefficient of each power of x lies in $V \otimes \mathbb{C}((t))$.

Let $v \in V$ and define the “generic vertex operator”

$$Y_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n) x^{-n-1} \in (V \otimes \mathbb{C}[t, t^{-1}])[[x, x^{-1}]]. \quad (5.4)$$

Then

$$\begin{aligned} Y_t(v, x) &= v \otimes x^{-1} \delta\left(\frac{t}{x}\right) \\ &= v \otimes t^{-1} \delta\left(\frac{x}{t}\right) \\ &\in V \otimes \mathbb{C}[[t, t^{-1}, x, x^{-1}]] \\ &(\subset V[[t, t^{-1}, x, x^{-1}]]), \end{aligned} \quad (5.5)$$

and the linear map

$$\begin{aligned} V &\rightarrow V \otimes \mathbb{C}[[t, t^{-1}, x, x^{-1}]] \\ v &\mapsto Y_t(v, x) \end{aligned} \quad (5.6)$$

is simply the map given by tensoring by the “universal element” $x^{-1} \delta\left(\frac{t}{x}\right)$. We have

$$\tau_W(Y_t(v, x)) = Y_W(v, x). \quad (5.7)$$

For all $f(x) \in \mathbb{C}[[x, x^{-1}]]$, $f(x)Y_t(v, x)$ is defined and

$$f(x)Y_t(v, x) = f(t)Y_t(v, x). \quad (5.8)$$

In case $f(x) \in \mathbb{C}((x))$, then $\tau_W(f(x)Y_t(v, x))$ is also defined, and

$$f(x)Y_W(v, x) = f(x)\tau_W(Y_t(v, x)) = \tau_W(f(x)Y_t(v, x)) = \tau_W(f(t)Y_t(v, x)). \quad (5.9)$$

The expansion coefficients, in powers of x , of $Y_t(v, x)$ span $v \otimes \mathbb{C}[t, t^{-1}]$, the x -expansion coefficients of $Y_W(v, x)$ span $\tau_W(v \otimes \mathbb{C}[t, t^{-1}])$ and for $f(x) \in \mathbb{C}[[x, x^{-1}]]$, the x -expansion coefficients of $f(x)Y_t(v, x)$ span $v \otimes f(t)\mathbb{C}[t, t^{-1}]$. In case $f(x) \in \mathbb{C}((x))$, the x -expansion coefficients of $f(x)Y_W(v, x)$ span $\tau_W(v \otimes f(t)\mathbb{C}[t, t^{-1}])$.

Using this viewpoint, we shall examine each of the three terms in the Jacobi identity (3.26) in the definition of logarithmic intertwining operator. First we consider the formal Laurent series in x_0, x_1, x_2 and t given by

$$\begin{aligned} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_t(v, x_0) &= x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) Y_t(v, x_0) \\ &= v \otimes x_1^{-1} \delta \left(\frac{x_2 + t}{x_1} \right) x_0^{-1} \delta \left(\frac{t}{x_0} \right) \end{aligned} \quad (5.10)$$

(cf. the right-hand side of (3.26)). The expansion coefficients in powers of x_0, x_1 and x_2 of (5.10) span just the space $v \otimes \mathbb{C}[t, t^{-1}]$. However, the expansion coefficients in x_0 and x_1 only (but not in x_2) of

$$\begin{aligned} x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) Y_t(v, x_0) &= v \otimes x_1^{-1} \delta \left(\frac{x_2 + t}{x_1} \right) x_0^{-1} \delta \left(\frac{t}{x_0} \right) \\ &= v \otimes \left(\sum_{m \in \mathbb{Z}} (x_2 + t)^m x_1^{-m-1} \right) \left(\sum_{n \in \mathbb{Z}} t^n x_0^{-n-1} \right) \end{aligned} \quad (5.11)$$

span

$$v \otimes \iota_{x_2, t} \mathbb{C}[t, t^{-1}, x_2 + t, (x_2 + t)^{-1}] \subset v \otimes \mathbb{C}[x_2, x_2^{-1}](t),$$

where $\iota_{x_2, t}$ is the operation of expanding a formal rational function in the indicated algebra as a formal Laurent series involving only finitely many negative powers of t (cf. the notation ι_{12} , etc., considered at the end of Section 2). We shall use similar ι -notations below. Specifically, the coefficient of $x_0^{-n-1} x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (5.11) is $v \otimes (x_2 + t)^m t^n$.

We may specialize $x_2 \mapsto z \in \mathbb{C}^\times$, and (5.11) becomes

$$\begin{aligned} z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) &= x_1^{-1} \delta \left(\frac{z + x_0}{x_1} \right) Y_t(v, x_0) \\ &= v \otimes x_1^{-1} \delta \left(\frac{z + t}{x_1} \right) x_0^{-1} \delta \left(\frac{t}{x_0} \right) \\ &= v \otimes \left(\sum_{m \in \mathbb{Z}} (z + t)^m x_1^{-m-1} \right) \left(\sum_{n \in \mathbb{Z}} t^n x_0^{-n-1} \right). \end{aligned} \quad (5.12)$$

The coefficient of $x_0^{-n-1} x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (5.12) is $v \otimes (z + t)^m t^n \in V \otimes \mathbb{C}((t))$, and these coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (z + t)^{-1}] \subset v \otimes \mathbb{C}((t)). \quad (5.13)$$

Our $Q(z)$ -tensor product construction in Subsection 5.3 below will be based on a certain action of the space $V \otimes \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$, and the description of this space as the span of the coefficients of the expression (5.12) (as $v \in V$ varies) will be very useful.

Now consider

$$\begin{aligned} x_0^{-1} \delta \left(\frac{-x_2 + x_1}{x_0} \right) Y_t(v, x_1) &= v \otimes x_0^{-1} \delta \left(\frac{-x_2 + t}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right) \\ &= v \otimes \left(\sum_{n \in \mathbb{Z}} (-x_2 + t)^n x_0^{-n-1} \right) \left(\sum_{m \in \mathbb{Z}} t^m x_1^{-m-1} \right) \end{aligned} \quad (5.14)$$

(cf. the second term on the left-hand side of (3.26)). The expansion coefficients in powers of x_0 and x_1 (but not x_2) span

$$v \otimes \iota_{x_2, t} \mathbb{C}[t, t^{-1}, -x_2 + t, (-x_2 + t)^{-1}],$$

and in fact the coefficient of $x_0^{-n-1} x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (5.14) is $v \otimes (-x_2 + t)^n t^m$. Again specializing $x_2 \mapsto z \in \mathbb{C}^\times$, we obtain

$$\begin{aligned} x_0^{-1} \delta \left(\frac{-z + x_1}{x_0} \right) Y_t(v, x_1) &= v \otimes x_0^{-1} \delta \left(\frac{-z + t}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right) \\ &= v \otimes \left(\sum_{n \in \mathbb{Z}} (-z + t)^n x_0^{-n-1} \right) \left(\sum_{m \in \mathbb{Z}} t^m x_1^{-m-1} \right). \end{aligned} \quad (5.15)$$

The coefficient of $x_0^{-n-1} x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (5.15) is $v \otimes (-z + t)^n t^m$, and these coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (-z + t)^{-1}] \subset v \otimes \mathbb{C}((t)). \quad (5.16)$$

Finally, consider

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_t(v, x_1) = v \otimes x_0^{-1} \delta \left(\frac{t - x_2}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right). \quad (5.17)$$

The coefficient of $x_0^{-n-1} x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) is $v \otimes (t - x_2)^n t^m$, and these expansion coefficients span

$$v \otimes \iota_{t, x_2} \mathbb{C}[t, t^{-1}, t - x_2, (t - x_2)^{-1}].$$

If we again specialize $x_2 \mapsto z$, we get

$$x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_t(v, x_1) = v \otimes x_0^{-1} \delta \left(\frac{t - z}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right), \quad (5.18)$$

whose coefficient of $x_0^{-n-1} x_1^{-m-1}$ is $v \otimes (t - z)^n t^m$. These coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (t - z)^{-1}] \subset v \otimes \mathbb{C}((t^{-1})) \quad (5.19)$$

(cf. (5.13), (5.16)).

In the construction of $P(z)$ -tensor products in Subsection 5.2, we shall also need the following expression, which is slightly different from what we have analyzed above:

$$x_0^{-1} \delta \left(\frac{x_1^{-1} - x_2}{x_0} \right) Y_t(v, x_1) = v \otimes x_0^{-1} \delta \left(\frac{t^{-1} - x_2}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right). \quad (5.20)$$

The coefficient of $x_0^{-n-1} x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) is $v \otimes (t^{-1} - x_2)^n t^m$, and these expansion coefficients span

$$v \otimes \iota_{t^{-1}, x_2} \mathbb{C}[t, t^{-1}, t^{-1} - x_2, (t^{-1} - x_2)^{-1}].$$

If we again specialize $x_2 \mapsto z$, we get

$$x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) = v \otimes x_0^{-1} \delta \left(\frac{t^{-1} - z}{x_0} \right) x_1^{-1} \delta \left(\frac{t}{x_1} \right), \quad (5.21)$$

whose coefficient of $x_0^{-n-1} x_1^{-m-1}$ is $v \otimes (t^{-1} - z)^n t^m$. These coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (t^{-1} - z)^{-1}] = v \otimes \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \subset v \otimes \mathbb{C}((t)). \quad (5.22)$$

Our $P(z)$ -tensor product construction in Subsection 5.2 below will be based on a certain action of the space $V \otimes \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$.

Later we shall evaluate the identity (3.26) on the elements of the contragredient module W'_3 . This will allow us to convert the expansion (5.19) into an expansion in positive powers of t . It will be useful to examine the notions of opposite and contragredient vertex operators more closely (recall Section 2, in particular, (2.57)).

We shall interpret the opposite vertex operator map Y_W^o by means of an operation on $V \otimes \mathbb{C}[[t, t^{-1}]]$ that will convert vertex operators into their opposites. We shall write this “opposite-operator” map, in various contexts, as “ o .” The operation o will be an involution. We proceed as follows: First we generalize Y^o in the following way: Recall that by Assumption 4.1, $L(1)$ acts nilpotently on any element $v \in V$. In particular, $e^{xL(1)}v$ is a polynomial in the formal variable x . Given any vector space U and any linear map

$$\begin{aligned} Z(\cdot, x) : V &\rightarrow U[[x, x^{-1}]] \quad \left(= \prod_{n \in \mathbb{Z}} U \otimes x^n \right) \\ v &\mapsto Z(v, x) \end{aligned} \quad (5.23)$$

from V into $U[[x, x^{-1}]]$ (i.e., given any family of linear maps from V into the spaces $U \otimes x^n$), we define $Z^o(\cdot, x) : V \rightarrow U[[x, x^{-1}]]$ by

$$Z^o(v, x) = Z(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}), \quad (5.24)$$

where we use the obvious linear map $Z(\cdot, x^{-1}) : V \rightarrow U[[x, x^{-1}]]$, and where we extend $Z(\cdot, x^{-1})$ canonically to a linear map $Z(\cdot, x^{-1}) : V[x, x^{-1}] \rightarrow U[[x, x^{-1}]]$. Then by formula (5.3.1) in [FHL] (the proof of Proposition 5.3.1), we have

$$\begin{aligned} Z^{oo}(v, x) &= Z^o(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\ &= Z(e^{x^{-1}L(1)}(-x^2)^{L(0)}e^{xL(1)}(-x^{-2})^{L(0)}v, x) \\ &= Z(v, x). \end{aligned} \quad (5.25)$$

That is,

$$Z^{oo}(\cdot, x) = Z(\cdot, x). \quad (5.26)$$

Moreover, if $Z(v, x) \in U((x))$, then $Z^o(v, x) \in U((x^{-1}))$ and vice versa.

Now we expand $Z(v, x)$ and $Z^o(v, x)$ in components. Write

$$Z(v, z) = \sum_{n \in \mathbb{Z}} v_{(n)} x^{-n-1}, \quad (5.27)$$

where for all $n \in \mathbb{Z}$,

$$\begin{aligned} V &\rightarrow U \\ v &\mapsto v_{(n)} \end{aligned} \quad (5.28)$$

is a linear map depending on $Z(\cdot, x)$ (and in fact, as $Z(\cdot, x)$ varies, these linear maps are arbitrary). Also write

$$Z^o(v, x) = \sum_{n \in \mathbb{Z}} v_{(n)}^o x^{-n-1} \quad (5.29)$$

where

$$\begin{aligned} V &\rightarrow U \\ v &\mapsto v_{(n)}^o \end{aligned} \quad (5.30)$$

is a linear map depending on $Z(\cdot, x)$. We shall compute $v_{(n)}^o$. First note that

$$\sum_{n \in \mathbb{Z}} v_{(n)}^o x^{-n-1} = \sum_{n \in \mathbb{Z}} (e^{xL(1)}(-x^{-2})^{L(0)}v)_{(n)} x^{n+1}. \quad (5.31)$$

For convenience, suppose that $v \in V_{(h)}$, for $h \in \mathbb{Z}$. Then the right-hand side of (5.31) is equal to

$$\begin{aligned} &(-1)^h \sum_{n \in \mathbb{Z}} (e^{xL(1)}v)_{(-n)} x^{-n+1-2h} \\ &= (-1)^h \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{(-n)} x^{m-n+1-2h} \\ &= (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} \sum_{n \in \mathbb{Z}} (L(1)^m v)_{(-n-m-2+2h)} x^{-n-1}, \end{aligned} \quad (5.32)$$

that is,

$$v_{(n)}^o = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{(-n-m-2+2h)}. \quad (5.33)$$

(Recall that by Assumption 4.1, $L(1)^m v = 0$ when m is sufficiently large, so that these expressions are well defined.) For $v \in V$ not necessarily homogeneous, $v_{(n)}^o$ is given by the appropriate sum of such expressions.

Now consider the special case where $U = V \otimes \mathbb{C}[t, t^{-1}]$ and where $Z(\cdot, x)$ is the “generic” linear map

$$\begin{aligned} Y_t(\cdot, x) : V &\rightarrow (V \otimes \mathbb{C}[t, t^{-1}])[[x, x^{-1}]] \\ v &\mapsto Y_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n) x^{-n-1} \end{aligned} \quad (5.34)$$

(recall (5.4)), i.e.,

$$v_{(n)} = v \otimes t^n. \quad (5.35)$$

Then for $v \in V_{(h)}$,

$$v_{(n)}^o = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} ((L(1))^m v) \otimes t^{-n-m-2+2h} \quad (5.36)$$

in this case.

This motivates defining an o -operation on $V \otimes \mathbb{C}[t, t^{-1}]$ as follows: For any $n, h \in \mathbb{Z}$ and $v \in V_{(h)}$, define

$$(v \otimes t^n)^o = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v) \otimes t^{-n-m-2+2h} \in V \otimes \mathbb{C}[t, t^{-1}], \quad (5.37)$$

and extend by linearity to $V \otimes \mathbb{C}[t, t^{-1}]$. That is, $(v \otimes t^n)^o = v_{(n)}^o$ for the special case $Z(\cdot, x) = Y_t(\cdot, x)$ discussed above. (Note that for general Z , we cannot expect to be able to define an analogous o -operation on U .) Also consider the map

$$\begin{aligned} Y_t^o(\cdot, x) &= (Y_t(\cdot, x))^o : V \rightarrow (V \otimes \mathbb{C}[t, t^{-1}])[x, x^{-1}] \\ v &\mapsto Y_t^o(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n)^o x^{-n-1}. \end{aligned} \quad (5.38)$$

Then for general $Z(\cdot, x)$ as above, we can define a linear map

$$\begin{aligned} \varepsilon_Z : V \otimes \mathbb{C}[t, t^{-1}] &\rightarrow U \\ v \otimes t^n &\mapsto v_{(n)} \end{aligned} \quad (5.39)$$

(“evaluation with respect to Z ”), i.e.,

$$\varepsilon_Z : Y_t(v, x) \mapsto Z(v, x), \quad (5.40)$$

and a linear map

$$\begin{aligned} \varepsilon_Z^o : V \otimes \mathbb{C}[t, t^{-1}] &\rightarrow U \\ v \otimes t^n &\mapsto v_{(n)}^o, \end{aligned} \quad (5.41)$$

i.e.,

$$\varepsilon_Z^o : Y_t^o(v, x) \mapsto Z^o(v, x). \quad (5.42)$$

Then

$$\varepsilon_Z^o = \varepsilon_Z \circ o, \quad (5.43)$$

that is,

$$\varepsilon_Z(Y_t^o(v, x)) = Z^o(v, x), \quad (5.44)$$

or equivalently, the diagram

$$\begin{array}{ccc} Y_t(v, x) & \xrightarrow{\varepsilon_Z} & Z(v, x) \\ o \downarrow & & \downarrow (Z(\cdot, x) \mapsto Z^o(\cdot, x)) \\ Y_t^o(v, x) & \xrightarrow{\varepsilon_Z} & Z^o(v, x) \end{array} \quad (5.45)$$

commutes. Note that the components $v_{(n)}^o$ of $Z^o(v, x)$ depend on all the components $v_{(n)}$ of $Z(v, z)$ (for arbitrary v), whereas the component $(v \otimes t^n)^o$ of $Y_t^o(v, z)$ can be defined generically and abstractly; $(v \otimes t^n)^o$ depends linearly on $v \in V$ alone.

Since in general $Z^{oo}(v, x) = Z(v, x)$, we know that

$$Y_t^{oo}(v, x) = Y_t(v, x) \quad (5.46)$$

as a special case, and in particular (and equivalently),

$$(v \otimes t^n)^{oo} = v \otimes t^n \quad (5.47)$$

for all $v \in V$ and $n \in \mathbb{Z}$. Thus o is an involution of $V \otimes \mathbb{C}[t, t^{-1}]$.

Furthermore, the involution o of $V \otimes \mathbb{C}[t, t^{-1}]$ extends canonically to a linear map

$$V \otimes \mathbb{C}[[t, t^{-1}]] \xrightarrow{o} V \otimes \mathbb{C}[[t, t^{-1}]].$$

In fact, consider the restriction of o to $V = V \otimes t^0$:

$$\begin{aligned} V &\xrightarrow{o} V \otimes \mathbb{C}[t, t^{-1}] \\ v &\mapsto v^o = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v) \otimes t^{-m-2+2h}, \end{aligned} \quad (5.48)$$

extended by linearity from $V_{(h)}$ to V . Then for $v \in V$, we may write

$$v^o = e^{t^{-1}L(1)}(-t^2)^{L(0)}vt^{-2}. \quad (5.49)$$

Also, for $v \in V$ and $n \in \mathbb{Z}$,

$$(v \otimes t^n)^o = v^o t^{-n}, \quad (5.50)$$

and it is clear that o extends to $V \otimes \mathbb{C}[[t, t^{-1}]]$: For $f(t) \in \mathbb{C}[[t, t^{-1}]]$,

$$(v \otimes f(t))^o = v^o f(t^{-1}). \quad (5.51)$$

To see that o is an involution of this larger space, first note that

$$v^{oo} = v \quad (5.52)$$

(although $v^o \notin V$ in general). (This could of course alternatively be proved by direct calculation using formula (5.37).) Also, for $g(t) \in \mathbb{C}[t, t^{-1}]$ and $f(t) \in \mathbb{C}[[t, t^{-1}]]$,

$$(v \otimes g(t)f(t))^o = v^o g(t^{-1})f(t^{-1}) = (v \otimes g(t))^o f(t^{-1}). \quad (5.53)$$

Thus for all $x \in V \otimes \mathbb{C}[t, t^{-1}]$ and $f(t) \in \mathbb{C}[[t, t^{-1}]]$,

$$(xf(t))^o = x^o f(t^{-1}). \quad (5.54)$$

It follows that

$$\begin{aligned}
(v \otimes f(t))^{oo} &= (v^o f(t^{-1}))^o \\
&= v^{oo} f(t) \\
&= v f(t) \\
&= v \otimes f(t),
\end{aligned} \tag{5.55}$$

and we have shown that o is an involution of $V \otimes \mathbb{C}[[t, t^{-1}]]$. We have

$$o : V \otimes \mathbb{C}((t)) \leftrightarrow V \otimes \mathbb{C}((t^{-1})). \tag{5.56}$$

Note that

$$\begin{aligned}
Y_t^o(v, x) &= \sum_{n \in \mathbb{Z}} (v \otimes t^n)^o x^{-n-1} \\
&= v^o \sum_{n \in \mathbb{Z}} t^{-n} x^{-n-1} \\
&= v^o x^{-1} \delta(tx) \\
&= v^o t \delta(tx) \\
&\in V \otimes \mathbb{C}[[t, t^{-1}, x, x^{-1}]].
\end{aligned} \tag{5.57}$$

Thus the map $v \mapsto Y_t^o(v, x)$ is the linear map given by multiplying v^o by the “universal element” $t\delta(tx)$ (cf. the comment following (5.6)). By (5.49), we also have

$$\begin{aligned}
Y_t^o(v, x) &= e^{t^{-1}L(1)}(-t^2)^{L(0)} v t^{-1} \delta(tx) \\
&= e^{xL(1)}(-x^{-2})^{L(0)} v \otimes x \delta(tx).
\end{aligned} \tag{5.58}$$

For all $f(x) \in \mathbb{C}[[x, x^{-1}]]$, $f(x)Y_t^o(v, x)$ is defined and

$$\begin{aligned}
f(x)Y_t^o(v, x) &= f(t^{-1})Y_t^o(v, x) \\
&= v^o f(t^{-1})t\delta(tx).
\end{aligned} \tag{5.59}$$

Now we return to the starting point—the original special case: $U = \text{End } W$ and $Z(\cdot, z) = Y_W(\cdot, z) : V \rightarrow (\text{End } W)[[x, x^{-1}]]$. The corresponding map

$$\begin{aligned}
\varepsilon_Z = \varepsilon_{Y_W} : V[t, t^{-1}] &\rightarrow \text{End } W \\
v \otimes t^n &\mapsto v_{(n)}
\end{aligned} \tag{5.60}$$

(recall (5.39)) is just the map $\tau_W : v \otimes t^n \mapsto v_n$ (recall (5.1)), i.e., $v_{(n)} = v_n$ in this case. Recall that this map extends canonically to $V \otimes \mathbb{C}((t))$. The map ε_Z^o is $\tau_W \circ o : V \otimes \mathbb{C}[t, t^{-1}] \rightarrow \text{End } W$ and this map extends canonically to $V \otimes \mathbb{C}((t^{-1}))$. In addition to (5.7), we have

$$\tau_W(Y_t^o(v, x)) = Y_W^o(v, x) \tag{5.61}$$

$(v_{(n)}^o = v_n^o$ in this case; recall (2.58)). In case $f(x) \in \mathbb{C}((x^{-1}))$,

$$f(x)Y_W^o(v, x) = \tau_W(f(x)Y_t^o(v, x))$$

is defined and is equal to $\tau_W(f(t^{-1})Y_t^o(v, z))$ (which is also defined).

The x -expansion coefficients of $f(x)Y_t^o(v, x)$, for $f(x) \in \mathbb{C}[[x, x^{-1}]]$, span

$$\begin{aligned} v^o f(t^{-1})\mathbb{C}[t, t^{-1}] &= (v\mathbb{C}[t, t^{-1}])^o f(t^{-1}) \\ &= (vf(t)\mathbb{C}[t, t^{-1}])^o. \end{aligned} \quad (5.62)$$

The x -expansion coefficients of $Y_W^o(v, x)$ span

$$\begin{aligned} \tau_W(v^o\mathbb{C}[t, t^{-1}]) &= \tau_W((v \otimes \mathbb{C}[t, t^{-1}])^o) \\ &= \tau_W^o(v \otimes \mathbb{C}[t, t^{-1}]). \end{aligned} \quad (5.63)$$

In case $f(x) \in \mathbb{C}((x^{-1}))$, the x -expansion coefficients of $f(x)Y_W^o(v, x)$ span

$$\tau_W(v^o f(t^{-1})\mathbb{C}[t, t^{-1}]) = \tau_W^o(vf(t)\mathbb{C}[t, t^{-1}]).$$

(Cf. the comments after (5.9).)

We shall need spaces of the forms $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ and $V \otimes \iota_- \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$, where we use the notations

$$\begin{aligned} \iota_+ : \mathbb{C}(t) &\hookrightarrow \mathbb{C}((t)) \subset \mathbb{C}[[t, t^{-1}]] \\ \iota_- : \mathbb{C}(t) &\hookrightarrow \mathbb{C}((t^{-1})) \subset \mathbb{C}[[t, t^{-1}]] \end{aligned} \quad (5.64)$$

to denote the operations of expanding a rational function of the formal variable t in the indicated directions (as in Section 8.1 of [FLM2]). We shall also need certain translation operations, as well as the o -operation. For $a \in \mathbb{C}$, we define the translation isomorphism

$$\begin{aligned} T_a : \mathbb{C}(t) &\xrightarrow{\sim} \mathbb{C}(t) \\ f(t) &\mapsto f(t+a) \end{aligned} \quad (5.65)$$

and (for our use below) we also set

$$T_a^\pm = \iota_\pm \circ T_a \circ \iota_\pm^{-1} : \iota_\pm \mathbb{C}(t) \hookrightarrow \mathbb{C}((t^{\pm 1})). \quad (5.66)$$

(Note that the domains of these maps consist of certain series expansions of formal rational functions rather than of formal rational functions themselves.) The following lemma will be needed for our action $\tau_{P(z)}$ in Subsection 5.2 below (we shall sometimes write $o(Y_t(v, x_1))$ for $Y_t^o(v, x_1)$, etc.):

Lemma 5.1 *Let $z \in \mathbb{C}^\times$. Then*

$$o \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) = x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t^o(v, x_1), \quad (5.67)$$

$$(\iota_+ \circ \iota_-^{-1} \circ o) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) = x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) Y_t^o(v, x_1), \quad (5.68)$$

$$\begin{aligned} & (\iota_+ \circ T_z \circ \iota_-^{-1} \circ o) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \\ &= z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) Y_t(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0). \end{aligned} \quad (5.69)$$

Proof Formula (5.67) is immediate from the definition of the map o (recall (5.37)). By (5.67), (5.57) and (2.5), we have

$$\begin{aligned} & (\iota_+ \circ \iota_-^{-1} \circ o) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \\ &= (\iota_+ \circ \iota_-^{-1}) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t^o(v, x_1) \right) \\ &= (\iota_+ \circ \iota_-^{-1}) \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) v^o t \delta(tx_1) \right) \\ &= (\iota_+ \circ \iota_-^{-1}) \left(x_0^{-1} \delta \left(\frac{t - z}{x_0} \right) v^o t \delta(tx_1) \right) \\ &= x_0^{-1} \delta \left(\frac{z - t}{-x_0} \right) v^o t \delta(tx_1) \\ &= x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) v^o t \delta(tx_1) \\ &= x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) Y_t^o(v, x_1), \end{aligned} \quad (5.70)$$

proving (5.68). For (5.69), note that by (5.58), the coefficient of x_0^{-n-1} in the right-hand side of (5.67) is

$$\begin{aligned} & (x_1^{-1} - z)^n \left(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v \otimes x_1 \delta \left(\frac{t}{x_1^{-1}} \right) \right) \\ &= (t - z)^n \left(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v \otimes x_1 \delta \left(\frac{t}{x_1^{-1}} \right) \right). \end{aligned}$$

Acted by $\iota_+ \circ T_z \circ \iota_-^{-1}$, this becomes

$$\begin{aligned} & t^n \left(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v \otimes x_1 \delta \left(\frac{z + t}{x_1^{-1}} \right) \right) \\ &= z^{-1} \delta \left(\frac{x_1^{-1} - t}{z} \right) (e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v \otimes t^n), \end{aligned}$$

which by (5.5) is the coefficient of x_0^{-n-1} in the right-hand side of (5.69). \square

We shall be interested in

$$T_{-z}^{\pm} : \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \hookrightarrow \mathbb{C}((t^{\pm})), \quad (5.71)$$

where z is an arbitrary nonzero complex number, as above. The images of these two maps are $\iota_{\pm} \mathbb{C}[t, t^{-1}, (z-t)^{-1}]$.

Extend the maps T_{-z}^{\pm} to linear isomorphisms

$$T_{-z}^{\pm} : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \xrightarrow{\sim} V \otimes \iota_{\pm} \mathbb{C}[t, t^{-1}, (z-t)^{-1}] \quad (5.72)$$

given by $1 \otimes T_{-z}^{\pm}$ with T_{-z}^{\pm} as defined above. Note that the domain of these two maps is described by (5.12)–(5.13), that the image of the map T_{-z}^+ is described by (5.15)–(5.16) and that the image of the map T_{-z}^- is described by (5.18)–(5.19).

We have the two mutually inverse maps

$$\begin{aligned} V \otimes \iota_- \mathbb{C}[t, t^{-1}, (z-t)^{-1}] &\xrightarrow{o} V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1}-t)^{-1}] \\ v \otimes f(t) &\mapsto v^o f(t^{-1}) \end{aligned} \quad (5.73)$$

and

$$\begin{aligned} V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1}-t)^{-1}] &\xrightarrow{o} V \otimes \iota_- \mathbb{C}[t, t^{-1}, (z-t)^{-1}] \\ v \otimes f(t) &\mapsto v^o f(t^{-1}), \end{aligned} \quad (5.74)$$

which are both isomorphisms. We form the composition

$$T_{-z}^o = o \circ T_{-z}^- \quad (5.75)$$

to obtain another isomorphism

$$T_{-z}^o : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \xrightarrow{\sim} V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1}-t)^{-1}].$$

The maps T_{-z}^+ and T_{-z}^o will be the main ingredients of our action $\tau_{Q(z)}$ (see Subsection 5.3 below). The following result asserts that T_{-z}^+ , T_{-z}^- and T_{-z}^o transform the expression (5.12) into (5.15), (5.18) and the o -transform of (5.18), respectively:

Lemma 5.2 *We have*

$$T_{-z}^+ \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) = x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) Y_t(v, x_1), \quad (5.76)$$

$$T_{-z}^- \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) = x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_t(v, x_1), \quad (5.77)$$

$$T_{-z}^o \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) = x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_t^o(v, x_1). \quad (5.78)$$

Proof We prove (5.76): From (5.12), the coefficient of $x_0^{-n-1}x_1^{-m-1}$ in the left-hand side of (5.76) is $T_{-z}^+(v \otimes (z+t)^m t^n)$. By the definitions,

$$T_{-z}^+(v \otimes (z+t)^m t^n) = v \otimes t^m(-(z-t))^n. \quad (5.79)$$

On the other hand, the right-hand side of (5.76) can be written as

$$v \otimes x_0^{-1} \delta\left(\frac{z-x_1}{-x_0}\right) x_1^{-1} \delta\left(\frac{t}{x_1}\right) = v \otimes x_0^{-1} \delta\left(\frac{z-t}{-x_0}\right) x_1^{-1} \delta\left(\frac{t}{x_1}\right), \quad (5.80)$$

where we have used (5.5) and the fundamental property (2.5) of the formal δ -function. The coefficient of $x_0^{-n-1}x_1^{-m-1}$ in the right-hand side of (5.80) is also $v \otimes t^m(-(z-t))^n$, proving (5.76). Formula (5.77) is proved similarly, and (5.78) is obtained from (5.77) by the application of the map o . \square

5.2 Constructions of $P(z)$ -tensor products

We proceed to the construction of $P(z)$ -tensor products. While one can certainly consider categories \mathcal{C} in Remark 4.22 that are not closed under the contragredient functor, it is most natural to consider such categories \mathcal{C} that are indeed closed under this functor (recall Notation 2.36). Our constructions of $P(z)$ -tensor products will in fact use the contragredient functor; the $P(z)$ -tensor product of (generalized) modules W_1 and W_2 will arise as the contragredient module of a certain subspace of the vector space dual $(W_1 \otimes W_2)^*$. We now present this “double-dual” approach to the construction of $P(z)$ -tensor products, generalizing the double-dual approach carried out in [HL5]–[HL7]. At first, we need not fix any subcategory \mathcal{C} of \mathcal{M}_{sg} or \mathcal{GM}_{sg} . As usual, we take $z \in \mathbb{C}^\times$.

We shall be constructing an action of the space $V \otimes \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on the space $(W_1 \otimes W_2)^*$, given generalized V -modules W_1 and W_2 . This action will be based on the translation operations and on the o -operation discussed in the preceding subsection. More precisely, it is the space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ whose action we shall define.

Let I be a $P(z)$ -intertwining map of type $\binom{W_3}{W_1 W_2}$, as in Definition 4.2. Consider the contragredient generalized V -module (W'_3, Y'_3) , recall the opposite vertex operator (2.57) and formula (2.73), and recall why the ingredients of formula (4.4) are well defined. For $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, applying $w'_{(3)}$ to (4.4), replacing x_1 by x_1^{-1} in the resulting formula and then replacing v by $e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v$, we get:

$$\begin{aligned} & \left\langle x_0^{-1} \delta\left(\frac{x_1^{-1} - z}{x_0}\right) Y'_3(v, x_1) w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \right\rangle \\ &= \left\langle w'_{(3)}, z^{-1} \delta\left(\frac{x_1^{-1} - x_0}{z}\right) I(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \right\rangle \\ &+ \left\langle w'_{(3)}, x_0^{-1} \delta\left(\frac{z - x_1^{-1}}{-x_0}\right) I(w_{(1)} \otimes Y_2^o(v, x_1) w_{(2)}) \right\rangle. \end{aligned} \quad (5.81)$$

We shall use this to motivate our action.

As we discussed in the preceding subsection (see (5.21) and (5.22)), in the left-hand side of (5.81), the coefficients of

$$x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_3'(v, x_1) \quad (5.82)$$

in powers of x_0 and x_1 , for all $v \in V$, span

$$\tau_{W_3}(V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]) \quad (5.83)$$

(recall (5.2) and (5.7)). Let us now define an action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on $(W_1 \otimes W_2)^*$.

Definition 5.3 Define the linear action $\tau_{P(z)}$ of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on $(W_1 \otimes W_2)^*$ by

$$\begin{aligned} (\tau_{P(z)}(\xi)\lambda)(w_{(1)} \otimes w_{(2)}) &= \lambda(\tau_{W_1}((\iota_+ \circ T_z \circ \iota_-^{-1} \circ o)\xi)w_{(1)} \otimes w_{(2)}) \\ &\quad + \lambda(w_{(1)} \otimes \tau_{W_2}((\iota_+ \circ \iota_-^{-1} \circ o)\xi)w_{(2)}) \end{aligned} \quad (5.84)$$

for $\xi \in V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. (The fact that the right-hand side is well defined follows immediately from the generating-function reformulation of (5.84) given in (5.86) below.) Denote by $Y'_{P(z)}$ the action of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2)^*$ thus defined, that is,

$$Y'_{P(z)}(v, x) = \tau_{P(z)}(Y_t(v, x)) \quad (5.85)$$

for $v \in V \otimes \mathbb{C}[t, t^{-1}]$.

By Lemma 5.1, (5.7) and (5.61), we see that formula (5.84) can be written in terms of generating functions as

$$\begin{aligned} & \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\ &\quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)}) \end{aligned} \quad (5.86)$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$; note that by (5.21)–(5.22), the expansion coefficients in x_0 and x_1 of the left-hand side span the space of elements in the left-hand side of (5.84). Compare formula (5.86) with the motivating formula (5.81). The generating function form of the action $Y'_{P(z)}$ can be obtained by taking Res_{x_0} of both sides of (5.86), that is,

$$\begin{aligned} (Y'_{P(z)}(v, x_1)\lambda)(w_{(1)} \otimes w_{(2)}) &= \lambda(w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)}) \\ &\quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}). \end{aligned} \quad (5.87)$$

Remark 5.4 Using the actions $\tau_{W'_3}$ and $\tau_{P(z)}$, we can write (5.81) as

$$\left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_3(v, x_1) w'_{(3)} \right) \circ I = \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) (w'_{(3)} \circ I)$$

or equivalently, as

$$\left(\tau_{W'_3} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) w'_{(3)} \right) \circ I = \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) (w'_{(3)} \circ I).$$

In the spirit of the discussion related to Lemma 4.36, we find it natural to introduce subspaces of $(W_1 \otimes W_2)^*$ homogeneous with respect to \tilde{A} . Since W_1 and W_2 are \tilde{A} -graded, $W_1 \otimes W_2$ also has a natural \tilde{A} -grading—the tensor product grading, and we shall write $(W_1 \otimes W_2)^{(\beta)}$ for the homogeneous subspace of degree $\beta \in \tilde{A}$ of $W_1 \otimes W_2$. For $\beta \in \tilde{A}$, let $((W_1 \otimes W_2)^*)^{(\beta)}$ be the space consisting of the elements $\lambda \in (W_1 \otimes W_2)^*$ such that $\lambda(\tilde{w}) = 0$ for $\tilde{w} \in (W_1 \otimes W_2)^{(\gamma)}$ with $\gamma \neq -\beta$. (Of course, the full space $(W_1 \otimes W_2)^*$ is not \tilde{A} -graded since it is not a direct sum of subspaces homogeneous with respect to \tilde{A} .)

The space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ also has an A -grading, induced from the A -grading on V : For $\alpha \in A$,

$$(V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}])^{(\alpha)} = V^{(\alpha)} \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]. \quad (5.88)$$

Using these gradings, we formulate:

Definition 5.5 We call a linear action τ of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on $(W_1 \otimes W_2)^*$ \tilde{A} -compatible if for $\alpha \in A$, $\beta \in \tilde{A}$, $\xi \in (V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}])^{(\alpha)}$ and $\lambda \in ((W_1 \otimes W_2)^*)^{(\beta)}$,

$$\tau(\xi)\lambda \in ((W_1 \otimes W_2)^*)^{(\alpha+\beta)}.$$

From (5.84) or (5.86), we have:

Proposition 5.6 *The action $\tau_{P(z)}$ is \tilde{A} -compatible.* \square

Remark 5.7 Notice that Proposition 5.6 is analogous to the condition (2.87) in the definition of the notion of (generalized) module. We now proceed to establish several more of the module-action properties for our action $\tau_{P(z)}$ on $(W_1 \otimes W_2)^*$, in both the conformal and Möbius cases. However, while we will prove the commutator formula for our action (see Proposition 5.9 below), we will *not* be able to prove the Jacobi identity on an element $\lambda \in (W_1 \otimes W_2)^*$ until we assume the “ $P(z)$ -compatibility condition” for the element λ (see Theorem 5.39 below). We shall be constructing a certain subspace $W_1 \boxtimes_{P(z)} W_2$ of $(W_1 \otimes W_2)^*$ which under suitable conditions will be a generalized V -module and whose contragredient module will be $W_1 \boxtimes_{P(z)} W_2$ (see Remark 5.27 and Proposition 5.32), and we shall use the $P(z)$ -compatibility condition to describe this subspace (see Theorem 5.45).

We have the following result generalizing Proposition 13.3 in [HL7]:

Proposition 5.8 *The action $Y'_{P(z)}$ has the property*

$$Y'_{P(z)}(\mathbf{1}, x) = 1,$$

where $\mathbf{1}$ on the right-hand side is the identity map of $(W_1 \otimes W_2)^*$. It also has the $L(-1)$ -derivative property

$$\frac{d}{dx} Y'_{P(z)}(v, x) = Y'_{P(z)}(L(-1)v, x)$$

for $v \in V$.

Proof The first statement follows directly from the definition. We prove the $L(-1)$ -derivative property. From (5.87), we obtain, using (2.62),

$$\begin{aligned} & \left(\frac{d}{dx} Y'_{P(z)}(v, x) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= \frac{d}{dx} \lambda(w_{(1)} \otimes Y_2^o(v, x) w_{(2)}) \\ & \quad + \frac{d}{dx} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ &= \lambda \left(w_{(1)} \otimes \frac{d}{dx} Y_2^o(v, x) w_{(2)} \right) \\ & \quad + \text{Res}_{x_0} \left(\frac{d}{dx} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ & \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \frac{d}{dx} \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ &= \lambda(w_{(1)} \otimes Y_2^o(L(-1)v, x) w_{(2)}) \\ & \quad + \text{Res}_{x_0} \left(\frac{d}{dx} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ & \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}L(1)(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ & \quad - 2 \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}L(0)x^{-1}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}). \quad (5.89) \end{aligned}$$

The second term on the right-hand side of (5.89) is equal to

$$\begin{aligned} & -\text{Res}_{x_0} x^{-2} \left(\frac{d}{dx^{-1}} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \right) \cdot \\ & \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{x_0} x^{-2} \left(\frac{d}{dx_0} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \right) \cdot \\ & \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0) w_{(1)} \otimes w_{(2)}) \end{aligned}$$

$$\begin{aligned}
&= -\text{Res}_{x_0} x^{-2} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\
&\quad \cdot \frac{d}{dx_0} \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
&= -\text{Res}_{x_0} x^{-2} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}). \tag{5.90}
\end{aligned}$$

By (5.90), (3.72) and (3.64), the right-hand side of (5.89) is equal to

$$\begin{aligned}
&\lambda(w_{(1)} \otimes Y_2^o(L(-1)v, x)w_{(2)}) \\
&\quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}L(-1)v, x_0)w_{(1)} \otimes w_{(2)}) \\
&= (Y'_{P(z)}(L(-1)v, x)\lambda)(w_{(1)} \otimes w_{(2)}),
\end{aligned}$$

proving the $L(-1)$ -derivative property. \square

Proposition 5.9 *The action $Y'_{P(z)}$ satisfies the commutator formula for vertex operators, that is, on $(W_1 \otimes W_2)^*$,*

$$\begin{aligned}
&[Y'_{P(z)}(v_1, x_1), Y'_{P(z)}(v_2, x_2)] \\
&= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{P(z)}(Y(v_1, x_0)v_2, x_2)
\end{aligned}$$

for $v_1, v_2 \in V$.

Proof In the following proof, the reader should note the well-definedness of each expression and the justifiability of each use of a δ -function property.

Let $\lambda \in (W_1 \otimes W_2)^*$, $v_1, v_2 \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. By (5.87),

$$\begin{aligned}
&(Y'_{P(z)}(v_1, x_1)Y'_{P(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= (Y'_{P(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2^o(v_1, x_1)w_{(2)}) \\
&\quad + \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) (Y'_{P(z)}(v_2, x_2)\lambda)(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)w_{(1)} \otimes w_{(2)}) \\
&= \lambda(w_{(1)} \otimes Y_2^o(v_2, x_2)Y_2^o(v_1, x_1)w_{(2)}) \\
&\quad + \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \lambda(Y_1(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)w_{(1)} \otimes Y_2^o(v_1, x_1)w_{(2)}) \\
&\quad + \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)w_{(1)} \otimes Y_2^o(v_2, x_2)w_{(2)}) \\
&\quad + \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)w_{(1)} \otimes w_{(2)}). \tag{5.91}
\end{aligned}$$

Transposing the subscripts 1 and 2 of the symbols v , x and y , we also have

$$\begin{aligned}
& (Y'_{P(z)}(v_2, x_2)Y'_{P(z)}(v_1, x_1)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda(w_{(1)} \otimes Y_2^o(v_1, x_1)Y_2^o(v_2, x_2)w_{(2)}) \\
&+ \text{Res}_{y_1} z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)w_{(1)} \otimes Y_2^o(v_2, x_2)w_{(2)}) \\
&+ \text{Res}_{y_2} z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)w_{(1)} \otimes Y_2^o(v_1, x_1)w_{(2)}) \\
&+ \text{Res}_{y_2} \text{Res}_{y_1} z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, y_1)Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)w_{(1)} \otimes w_{(2)}). \quad (5.92)
\end{aligned}$$

The equalities (5.91) and (5.92) give

$$\begin{aligned}
& ([Y'_{P(z)}(v_1, x_1), Y'_{P(z)}(v_2, x_2)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda(w_{(1)} \otimes [Y_2^o(v_2, x_2), Y_2^o(v_1, x_1)]w_{(2)}) \\
&\quad - \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) \cdot \\
&\quad \cdot \lambda([Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, y_1), Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)]w_{(1)} \otimes w_{(2)}) \\
&= \text{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0)v_2, x_2)w_{(2)}) \\
&\quad - \text{Res}_{y_1} \text{Res}_{y_2} \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \cdot \\
&\quad \cdot \lambda(Y_1(Y(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v_1, x_0)e^{x_2 L(1)}(-x_2^{-2})^{L(0)}v_2, y_2)w_{(1)} \otimes w_{(2)}) \quad (5.93)
\end{aligned}$$

(recall (2.61)).

But we have

$$\begin{aligned}
& z^{-1} \delta\left(\frac{x_1^{-1} - y_1}{z}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= \left(\sum_{m, n \in \mathbb{Z}} \frac{(x_1^{-1} - y_1)^m}{z^{m+1}} \frac{(x_2^{-1} - y_2)^n}{z^{n+1}} \right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= \left(\sum_{m, n \in \mathbb{Z}} (x_2^{-1} - y_2)^{-1} \left(\frac{x_1^{-1} - y_1}{x_2^{-1} - y_2}\right)^m \frac{(x_2^{-1} - y_2)^{m+n+1}}{z^{m+n+2}} \right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= \left(\sum_{m, k \in \mathbb{Z}} (x_2^{-1} - y_2)^{-1} \left(\frac{x_1^{-1} - y_1}{x_2^{-1} - y_2}\right)^m z^{-1} \left(\frac{x_2^{-1} - y_2}{z}\right)^k \right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= (x_2^{-1} - y_2)^{-1} \delta\left(\frac{x_1^{-1} - y_1}{x_2^{-1} - y_2}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) y_2^{-1} \delta\left(\frac{y_1 - x_0}{y_2}\right) \\
&= x_2 \delta\left(\frac{x_1^{-1} - (y_1 - y_2)}{x_2^{-1}}\right) z^{-1} \delta\left(\frac{x_2^{-1} - y_2}{z}\right) y_1^{-1} \delta\left(\frac{y_2 + x_0}{y_1}\right)
\end{aligned}$$

$$= x_2 \delta \left(\frac{x_1^{-1} - x_0}{x_2^{-1}} \right) z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right). \quad (5.94)$$

By (3.59), (3.60) and (3.65), we also have

$$\begin{aligned} & Y(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, x_0) e^{x_2 L(1)}(-x_2^{-2})^{L(0)} \\ &= e^{x_2 L(1)} Y \left(e^{-x_2(1+x_0 x_2) L(1)} (1+x_0 x_2)^{-2L(0)} e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, \frac{x_0}{1+x_0 x_2} \right) (-x_2^{-2})^{L(0)} \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left((-x_2^2)^{L(0)} e^{-x_2(1+x_0 x_2) L(1)} \cdot \right. \\ &\quad \left. \cdot (1+x_0 x_2)^{-2L(0)} e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, -\frac{x_0 x_2^2}{1+x_0 x_2} \right) \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{-x_2(1+x_0 x_2)(-x_2^{-2}) L(1)} (-x_2^2)^{L(0)} \cdot \right. \\ &\quad \left. \cdot (1+x_0 x_2)^{-2L(0)} e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, -\frac{x_0 x_2^2}{1+x_0 x_2} \right) \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{(x_2^{-1}+x_0) L(1)} (-(x_2^{-1}+x_0)^2)^{-L(0)} e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v_1, -\frac{x_0 x_2}{x_2^{-1}+x_0} \right) \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{(x_2^{-1}+x_0) L(1)} e^{-x_1(x_2^{-1}+x_0)^2 L(1)} \cdot \right. \\ &\quad \left. \cdot (-(x_2^{-1}+x_0)^2)^{-L(0)} (-x_1^{-2})^{L(0)} v_1, -\frac{x_0 x_2}{x_2^{-1}+x_0} \right) \\ &= e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{(x_2^{-1}+x_0) L(1)} e^{-x_1(x_2^{-1}+x_0)^2 L(1)} ((x_2^{-1}+x_0)x_1)^{-2L(0)} v_1, -\frac{x_0 x_2}{x_2^{-1}+x_0} \right). \end{aligned} \quad (5.95)$$

Using (5.94), (5.95) and the basic properties of the formal delta function, we see that (5.93) becomes

$$\begin{aligned} & ([Y'_{P(z)}(v_1, x_1), Y'_{P(z)}(v_2, x_2)] \lambda)(w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \\ &\quad - \text{Res}_{y_1} \text{Res}_{y_2} \text{Res}_{x_0} x_2 \delta \left(\frac{x_1^{-1} - x_0}{x_2^{-1}} \right) z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right) \cdot \\ &\quad \cdot \lambda \left(Y_1 \left(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y \left(e^{(x_2^{-1}+x_0) L(1)} e^{-x_1(x_2^{-1}+x_0)^2 L(1)} \cdot \right. \right. \right. \\ &\quad \left. \left. \left. \cdot ((x_2^{-1}+x_0)x_1)^{-2L(0)} v_1, -\frac{x_0 x_2}{x_2^{-1}+x_0} \right) v_2, y_2 \right) w_{(1)} \otimes w_{(2)} \right) \\ &= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{x_0} \text{Res}_{y_2} x_2 \delta \left(\frac{x_1^{-1} - x_0}{x_2^{-1}} \right) z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y(e^{x_1^{-1} L(1)} e^{-x_1^{-1} L(1)} \cdot (x_1^{-1} x_1)^{-2L(0)} v_1, -x_0 x_1 x_2) v_2, y_2) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \\
& \quad - \text{Res}_{x_0} x_2 \delta \left(\frac{x_2 + (-x_0 x_1 x_2)}{x_1} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y(v_1, -x_0 x_1 x_2) v_2, y_2) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \\
& \quad + \text{Res}_{y_0} x_1^{-1} \delta \left(\frac{x_2 + y_0}{x_1} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y(v_1, y_0) v_2, y_2) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v_1, x_0) v_2, x_2) w_{(2)}) \\
& \quad + \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} Y(v_1, x_0) v_2, y_2) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (Y'_{P(z)}(Y(v_1, x_0) v_2, x_2) \lambda)(w_{(1)} \otimes w_{(2)}). \tag{5.96}
\end{aligned}$$

Since λ , $w_{(1)}$ and $w_{(2)}$ are arbitrary, this equality gives the commutator formula for $Y'_{P(z)}$. \square

The following observations are analogous to those in Remark 8.1 of [HL6] (concerning the case of $Q(z)$ rather than $P(z)$):

Remark 5.10 The proof of Proposition 5.9 suggests the following: Using the definitions (5.84) and (5.86) as motivation, we define a (linear) action $\sigma_{P(z)}$ of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on the vector space $W_1 \otimes W_2$ (as opposed to $(W_1 \otimes W_2)^*$) as follows:

$$\sigma_{P(z)}(\xi)(w_{(1)} \otimes w_{(2)}) = \tau_{W_1}((\iota_+ \circ T_z \circ \iota_-^{-1} \circ o)\xi) w_{(1)} \otimes w_{(2)} + w_{(1)} \otimes \tau_{W_2}((\iota_+ \circ \iota_-^{-1} \circ o)\xi) w_{(2)} \tag{5.97}$$

for $\xi \in V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, or equivalently,

$$\begin{aligned}
& \left(\sigma_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \right) (w_{(1)} \otimes w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)} \\
& \quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) w_{(1)} \otimes Y_2^o(v, x_1) w_{(2)}. \tag{5.98}
\end{aligned}$$

That is, the operators $\sigma_{P(z)}(\xi)$ and $\tau_{P(z)}(\xi)$ are mutually adjoint:

$$(\tau_{P(z)}(\xi)\lambda)(w_{(1)} \otimes w_{(2)}) = \lambda(\sigma_{P(z)}(\xi)(w_{(1)} \otimes w_{(2)})). \quad (5.99)$$

While this action on $W_1 \otimes W_2$ is not very useful, it has the following three properties:

$$\sigma_{P(z)}(Y_t(\mathbf{1}, x)) = 1, \quad (5.100)$$

$$\frac{d}{dx} \sigma_{P(z)}(Y_t(v, x)) = \sigma_{P(z)}(Y_t(L(-1)v, x)) \quad (5.101)$$

for $v \in V$,

$$\begin{aligned} & [\sigma_{P(z)}(Y_t(v_2, x_2)), \sigma_{P(z)}(Y_t(v_1, x_1))] \\ &= \text{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \sigma_{P(z)}(Y_t(Y(v_1, x_0)v_2, x_2)) \end{aligned} \quad (5.102)$$

for $v_1, v_2 \in V$ (the opposite commutator formula). These follow either from the assertions of Propositions 5.8 and 5.9 or, better, from the fact that it was actually (5.100)–(5.102) that the proofs of these propositions were proving.

Remark 5.11 Taking Res_{x_0} of (5.98), we obtain

$$\begin{aligned} & (\sigma_{P(z)}(Y_t(v, x_1)))(w_{(1)} \otimes w_{(2)}) \\ &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1^{-1} - x_0}{z}\right) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)} \\ & \quad + w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)}. \end{aligned} \quad (5.103)$$

Substituting first $(-x_1^{-2})^{-L(0)}e^{-x_1 L(1)}v$ for v in (5.103) and then x_1^{-1} for x_1 in the same formula and using (3.65), (5.5) and (5.58), we obtain

$$\begin{aligned} (\sigma_{P(z)}(Y_t^o(v, x_1)))(w_{(1)} \otimes w_{(2)}) &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y_1(v, x_0)w_{(1)} \otimes w_{(2)} \\ & \quad + w_{(1)} \otimes Y_2(v, x_1)w_{(2)}. \end{aligned} \quad (5.104)$$

Using this, we see that $\sigma_{P(z)}$ can actually be viewed as a map from $V[t, t^{-1}]$ to $V((t)) \otimes V[t, t^{-1}]$ defined by

$$\begin{aligned} \sigma_{P(z)}(Y_t^o(v, x_1)) &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0) \otimes \mathbf{1} \\ & \quad + \mathbf{1} \otimes Y_t(v, x_1). \end{aligned} \quad (5.105)$$

Let $\Delta_{P(z)} = \sigma_{P(z)} \circ o$. Then (5.105) becomes

$$\begin{aligned} \Delta_{P(z)}(Y_t(v, x_1)) &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0) \otimes \mathbf{1} \\ & \quad + \mathbf{1} \otimes Y_t(v, x_1), \end{aligned} \quad (5.106)$$

which again can be viewed as a map

$$\Delta_{P(z)} : V[t, t^{-1}] \rightarrow V((t)) \otimes V[t, t^{-1}]. \quad (5.107)$$

In formula (2.4) of [MS], Moore and Seiberg introduced a map $\Delta_{z,0}$ which in fact corresponds exactly to the map $\Delta_{P(z)}$ defined by (5.106). They proposed to define a V -module structure (called “a representation of \mathcal{A} ” in [MS], where \mathcal{A} corresponds to our vertex algebra V) on $W_1 \otimes W_2$ by using this map, which can be viewed as a sort of analogue of a coproduct, but they acknowledged that E. Witten pointed out “subtleties in this definition which are related to the fact that [a] representation of \mathcal{A} obtained this way is not always a highest weight representation.” In fact, it is these subtleties that make it impossible to work with $W_1 \otimes W_2$; in virtually all interesting cases, $W_1 \otimes W_2$ does not have a natural (generalized) V -module structure. This is exactly the reason why we had to use a completely different approach to construct our $P(z)$ -tensor product of W_1 and W_2 .

When V is in fact a conformal (rather than Möbius) vertex algebra, we will write

$$Y'_{P(z)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{P(z)}(n) x^{-n-2}. \quad (5.108)$$

Then from the last two propositions we see that the coefficient operators of $Y'_{P(z)}(\omega, x)$ satisfy the Virasoro algebra commutator relations, that is,

$$[L'_{P(z)}(m), L'_{P(z)}(n)] = (m - n)L'_{P(z)}(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c,$$

with $c \in \mathbb{C}$ the central charge of V (recall Definition 2.2). Moreover, in this case, by setting $v = \omega$ in (5.87) and taking the coefficient of x_1^{-j-2} for $j = -1, 0, 1$, we find that

$$\begin{aligned} & (L'_{P(z)}(j)\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= \lambda \left(w_{(1)} \otimes L(-j)w_{(2)} + \left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes w_{(2)} \right), \end{aligned} \quad (5.109)$$

by (2.63). If V is just a Möbius vertex algebra, we define the actions $L'_{P(z)}(j)$ on $(W_1 \otimes W_2)^*$ by (5.109) for $j = -1, 0$ and 1 .

Remark 5.12 In view of the action $L'_{P(z)}(j)$, the $\mathfrak{sl}(2)$ -bracket relations (4.5) for a $P(z)$ -intertwining map I , with notation as in Definition 4.2, can be written as

$$(L'(j)w'_{(3)}) \circ I = L'_{P(z)}(j)(w'_{(3)} \circ I) \quad (5.110)$$

for $w'_{(3)} \in W'_3$ and $j = -1, 0$ and 1 (cf. (5.81) and Remark 5.4).

Remark 5.13 We have

$$L'_{P(z)}(j)((W_1 \otimes W_2)^*)^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

for $j = -1, 0, 1$ and $\beta \in \tilde{A}$ (cf. Proposition 5.6).

When V is a conformal vertex algebra, from the commutator formula for $Y'_{P(z)}(\omega, x)$, we see that $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$ realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$ (recall (2.27)) on $(W_1 \otimes W_2)^*$. When V is just a Möbius vertex algebra, the same conclusion still holds but a proof is needed. We state this as a proposition:

Proposition 5.14 *Let V be a Möbius vertex algebra and let W_1 and W_2 be generalized V -modules. Then the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$ realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$ on $(W_1 \otimes W_2)^*$, according to (2.27).*

Proof For $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $j, k = -1, 0, 1$, we have

$$\begin{aligned}
& (L'_{P(z)}(j)L'_{P(z)}(k)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= (L'_{P(z)}(k)\lambda) \left(w_{(1)} \otimes L(-j)w_{(2)} + \left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= (L'_{P(z)}(k)\lambda)(w_{(1)} \otimes L(-j)w_{(2)}) \\
&\quad + (L'_{P(z)}(k)\lambda) \left(\left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= \lambda \left(w_{(1)} \otimes L(-k)L(-j)w_{(2)} + \left(\sum_{l=0}^{1-k} \binom{1-k}{l} z^l L(-k-l) \right) w_{(1)} \otimes L(-j)w_{(2)} \right) \\
&\quad + \lambda \left(\left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes L(-k)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-k} \binom{1-k}{l} z^l L(-k-l) \right) \left(\sum_{i=0}^{1-j} \binom{1-j}{i} z^i L(-j-i) \right) w_{(1)} \otimes w_{(2)} \right). \tag{5.111}
\end{aligned}$$

From formula (5.111) we obtain

$$\begin{aligned}
& ([L'_{P(z)}(j), L'_{P(z)}(k)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda \left(w_{(1)} \otimes [L(-k), L(-j)]w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-k} \sum_{i=0}^{1-j} \binom{1-k}{l} \binom{1-j}{i} z^{l+i} [L(-k-l), L(-j-i)] \right) w_{(1)} \otimes w_{(2)} \right) \\
&= \lambda \left(w_{(1)} \otimes (j-k)L(-k-j)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-k} \sum_{i=0}^{1-j} \binom{1-k}{l} \binom{1-j}{i} z^{l+i} (j+i-k-l)L(-k-l-j-i) \right) w_{(1)} \otimes w_{(2)} \right) \tag{5.112}
\end{aligned}$$

Taking $j = 1$ and $k = -1, 0$ in (5.112), we obtain

$$\begin{aligned}
& ([L'_{P(z)}(1), L'_{P(z)}(k)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda \left(w_{(1)} \otimes (1-k)L(-k-1)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-k} \binom{1-k}{l} z^l (1-k-l)L(-k-l-1) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= \lambda \left(w_{(1)} \otimes (1-k)L(-k-1)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-(k+1)} \binom{1-k}{l} z^l (1-k-l)L(-k-l-1) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= (1-k)\lambda \left(w_{(1)} \otimes L(-1-k)w_{(2)} \right. \\
&\quad \left. + \left(\sum_{l=0}^{1-(k+1)} \binom{1-(k+1)}{l} z^l L(-(k+1)-l) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= ((1-k)L'_{P(z)}(1+k)\lambda)(w_{(1)} \otimes w_{(2)}),
\end{aligned}$$

proving the commutator formula

$$[L'_{P(z)}(1), L'_{P(z)}(k)] = (1-k)L'_{P(z)}(1+k)$$

for $k = -1, 0$.

Taking $j = 0$ and $k = -1$ in (5.112), we obtain

$$\begin{aligned}
& ([L'_{P(z)}(0), L'_{P(z)}(-1)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \lambda \left(w_{(1)} \otimes L(1)w_{(2)} + \left(\sum_{l=0}^2 \sum_{i=0}^1 \binom{2}{l} z^{l+i} (i+1-l)L(1-l-i) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= \lambda(w_{(1)} \otimes L(1)w_{(2)} + (L(1) + 2zL(0) + z^2L(-1))w_{(1)} \otimes w_{(2)}) \\
&= \lambda \left(w_{(1)} \otimes L(1)w_{(2)} + \left(\sum_{m=0}^{1-(-1)} \binom{2}{m} z^m L(-(-1)-m) \right) w_{(1)} \otimes w_{(2)} \right) \\
&= (L'_{P(z)}(-1)\lambda)(w_{(1)} \otimes w_{(2)}),
\end{aligned}$$

proving that

$$[L'_{P(z)}(0), L'_{P(z)}(-1)] = L'_{P(z)}(-1).$$

The three commutator formulas we have proved show that $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$ indeed realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$. \square

The commutator formulas corresponding to (2.28)–(2.30) (recall Definition 2.11)) also need to be proved in the Möbius case:

Proposition 5.15 *Let V be a Möbius vertex algebra and let W_1 and W_2 be generalized V -modules. Then for $v \in V$, we have the following commutator formulas:*

$$[L(-1), Y'_{P(z)}(v, x)] = Y'_{P(z)}(L(-1)v, x), \quad (5.113)$$

$$[L(0), Y'_{P(z)}(v, x)] = Y'_{P(z)}(L(0)v, x) + xY'_{P(z)}(L(-1)v, x), \quad (5.114)$$

$$[L(1), Y'_{P(z)}(v, x)] = Y'_{P(z)}(L(1)v, x) + 2xY'_{P(z)}(L(0)v, x) + x^2Y'_{P(z)}(L(-1)v, x), \quad (5.115)$$

where for brevity we write $L'_{P(z)}(j)$ acting on $(W_1 \otimes W_2)^*$ as $L(j)$.

Proof Let $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Using (5.109), (5.87), the commutator formulas for $L(j)$ and $Y_1(v, x_0)$ for $j = -1, 0, 1$ and $v \in V$ (recall Definition 2.11), and the commutator formulas for $L(j)$ and $Y_2^o(v, x)$ for $j = -1, 0, 1$ and $v \in V$ (recall Lemma 2.22), we obtain, for $j = -1, 0, 1$,

$$\begin{aligned} & ([L(j), Y'_{P(z)}(v, x)]\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= (L(j)Y'_{P(z)}(v, x)\lambda)(w_{(1)} \otimes w_{(2)}) - (Y'_{P(z)}(v, x)L(j)\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= (Y'_{P(z)}(v, x)\lambda)(w_{(1)} \otimes L(-j)w_{(2)}) \\ &\quad + \sum_{i=0}^{1-j} \binom{1-j}{i} (Y'_{P(z)}(v, x)\lambda)(z^i L(-j-i)w_{(1)} \otimes w_{(2)}) \\ &\quad - (L(j)\lambda)(w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\ &\quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) (L(j)\lambda)(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\ &= \lambda(w_{(1)} \otimes Y_2^o(v, x)L(-j)w_{(2)}) \\ &\quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes L(-j)w_{(2)}) \\ &\quad + \sum_{i=0}^{1-j} \binom{1-j}{i} \lambda(z^i L(-j-i)w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\ &\quad + \sum_{i=0}^{1-j} \binom{1-j}{i} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\ &\quad \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)z^i L(-j-i)w_{(1)} \otimes w_{(2)}) \\ &\quad - \lambda(w_{(1)} \otimes L(-j)Y_2^o(v, x)w_{(2)}) \\ &\quad - \sum_{i=0}^{1-j} \binom{1-j}{i} \lambda(z^i L(-j-i)w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\ &\quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes L(-j)w_{(2)}) \\ &\quad - \sum_{i=0}^{1-j} \binom{1-j}{i} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \end{aligned}$$

$$\begin{aligned}
& \cdot \lambda(z^i L(-j-i) Y_1(e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& = \lambda(w_{(1)} \otimes [Y_2^o(v, x), L(-j)] w_{(2)}) \\
& \quad - \sum_{i=0}^{1-j} \binom{1-j}{i} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\
& \quad \cdot \lambda([z^i L(-j-i), Y_1(e^{xL(1)}(-x^{-2})^{L(0)} v, x_0)] w_{(1)} \otimes w_{(2)}) \\
& = \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} \lambda(w_{(1)} \otimes Y_2^o(L(k-1)v, x) w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \sum_{i=0}^{1-j} \sum_{k=0}^{-j-i+1} \binom{1-j}{i} \binom{-j-i+1}{k} z^i x_0^{-j-i+1-k} \cdot \\
& \quad \cdot \lambda(Y_1(L(k-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}). \tag{5.116}
\end{aligned}$$

Using (2.6) and (2.11) when necessary, we see that the second term on the right-hand side of (5.116) is equal to the following expressions for $j = 1, 0$ and -1 , respectively:

$$\begin{aligned}
& - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}), \tag{5.117} \\
& - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \sum_{i=0}^1 \sum_{k=0}^{-i+1} \binom{1}{i} \binom{-j-i+1}{k} z^i x_0^{-j-i+1-k} \cdot \\
& \quad \cdot \lambda(Y_1(L(k-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& = - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x_0 \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) z \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& = - \text{Res}_{x_0} x \delta \left(\frac{z + x_0}{x^{-1}} \right) (x_0 + z) \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& = - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-1} \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \tag{5.118}
\end{aligned}$$

and

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \sum_{i=0}^2 \sum_{k=0}^{2-i} \binom{2}{i} \binom{2-i}{k} z^i x_0^{2-i-k} \\
& \quad \cdot \lambda(Y_1(L(k-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x_0^2 \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2x_0 \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2zx_0 \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2z \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) z^2 \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-2} \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2x^{-1} \lambda(Y_1(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(L(1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}).
\end{aligned} \tag{5.119}$$

Using (3.72) and (3.64), we see that (5.117), (5.118) and (5.119) are respectively equal to

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \lambda(Y_1(e^{xL(1)}(x^2L(1) - 2xL(0) + L(-1))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(L(1) + 2xL(0) + x^2L(-1))v, x_0)w_{(1)} \otimes w_{(2)}),
\end{aligned} \tag{5.120}$$

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-1} \cdot \lambda(Y_1(e^{xL(1)}(x^2L(1) - 2xL(0) + L(-1))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)})
\end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-xL(1) + L(0))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-1} \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(L(1) + 2xL(0) + x^2L(-1))v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(-x^{-1}L(1) - L(0))v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(L(0) + xL(-1))v, x_0)w_{(1)} \otimes w_{(2)})
\end{aligned} \tag{5.121}$$

and

$$\begin{aligned}
& -\text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-2} \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(x^2L(1) - 2xL(0) + L(-1))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2x^{-1} \lambda(Y_1(e^{xL(1)}(-xL(1) + L(0))(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& - \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}L(1)(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) x^{-2} \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(L(1) + 2xL(0) + x^2L(-1))v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) 2x^{-1} \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}(-x^{-1}L(1) - L(0))v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}x^{-2}L(1)v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}L(-1)v, x_0)w_{(1)} \otimes w_{(2)}).
\end{aligned} \tag{5.122}$$

The right-hand sides of (5.120), (5.121) and (5.122) can be written as

$$\sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}L(k-1)v, x_0)w_{(1)} \otimes w_{(2)}),$$

for $j = 1, 0, -1$, respectively. Thus the right-hand side of (5.116) is equal to

$$\sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} \lambda(w_{(1)} \otimes Y_2^o(L(k-1)v, x)w_{(2)})$$

$$\begin{aligned}
& + \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} \text{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)} L(k-1)v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} (Y'_{P(z)}(L(k-1)v, x)\lambda)(w_{(1)} \otimes w_{(2)}), \tag{5.123}
\end{aligned}$$

proving the proposition. \square

We have seen in (5.2), (5.82) and (5.83) that for a generalized V -module (W, Y_W) , the space $V \otimes \mathbb{C}((t))$, and in particular, the space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$, acts naturally on W via the action τ_W , in view of (2.49) and Assumption 4.1; recall that $v \otimes t^n$ ($v \in V$, $n \in \mathbb{Z}$) acts as the component v_n of $Y_W(v, x)$, and that more generally,

$$\tau_W \left(v \otimes \sum_{n>N} a_n t^n \right) = \sum_{n>N} a_n v_n \tag{5.124}$$

for $a_n \in \mathbb{C}$. For generalized V -modules W_1 , W_2 and W_3 , we shall next relate the $P(z)$ -intertwining maps of type $\binom{W_3}{W_1 W_2}$ to certain linear maps from W'_3 to $(W_1 \otimes W_2)^*$ intertwining the actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ and of $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$ (see Proposition 5.22 and Notation 5.23 below). For this, as is suggested by Lemma 4.36 and Proposition 5.6, we need to consider \tilde{A} -compatibility for linear maps from W_3 to $(W_1 \otimes W_2)^*$:

Definition 5.16 We call a map $J \in \text{Hom}(W_3, (W_1 \otimes W_2)^*)$ \tilde{A} -compatible if

$$J((W_3)^{(\beta)}) \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

for $\beta \in \tilde{A}$.

As in the discussion preceding Lemma 4.36, we see that an element λ of $(W_1 \otimes W_2 \otimes W_3)^*$ amounts exactly to a linear map

$$J_\lambda : W_3 \rightarrow (W_1 \otimes W_2)^*.$$

If λ is \tilde{A} -compatible (see that discussion), then for $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$ and $w_{(3)} \in W_3^{(\delta)}$ such that $\beta + \gamma + \delta \neq 0$,

$$J_\lambda(w_{(3)})(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0,$$

so that

$$J_\lambda(w_{(3)}) \in ((W_1 \otimes W_2)^*)^{(\delta)},$$

and so J_λ is \tilde{A} -compatible. Similarly, if J_λ is \tilde{A} -compatible, then so is λ . Thus using Lemma 4.36 we have:

Lemma 5.17 *The linear functional $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ is \tilde{A} -compatible if and only if J_λ is \tilde{A} -compatible. The map given by $\lambda \mapsto J_\lambda$ is the unique linear isomorphism from the space of \tilde{A} -compatible elements of $(W_1 \otimes W_2 \otimes W_3)^*$ to the space of \tilde{A} -compatible linear maps from W_3 to $(W_1 \otimes W_2)^*$ such that*

$$J_\lambda(w_{(3)})(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. In particular, the correspondence $I_\lambda \mapsto J_\lambda$ defines a (unique) linear isomorphism from the space of \tilde{A} -compatible linear maps

$$I = I_\lambda : W_1 \otimes W_2 \rightarrow \overline{W'_3}$$

to the space of \tilde{A} -compatible linear maps

$$J = J_\lambda : W_3 \rightarrow (W_1 \otimes W_2)^*$$

such that

$$\langle w_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle = J(w_{(3)})(w_{(1)} \otimes w_{(2)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. \square

Remark 5.18 From Lemma 5.17 (with W_3 replaced by W'_3) we have a canonical isomorphism from the space of \tilde{A} -compatible linear maps

$$I : W_1 \otimes W_2 \rightarrow \overline{W'_3}$$

to the space of \tilde{A} -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*,$$

determined by:

$$\langle w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle = J(w'_{(3)})(w_{(1)} \otimes w_{(2)}) \quad (5.125)$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, or equivalently,

$$w'_{(3)} \circ I = J(w'_{(3)}) \quad (5.126)$$

for $w'_{(3)} \in W'_3$.

We introduce another notion, corresponding to the lower truncation condition (4.3) for $P(z)$ -intertwining maps:

Definition 5.19 We call a map $J \in \text{Hom}(W_3, (W_1 \otimes W_2)^*)$ *grading restricted* if for $n \in \mathbb{C}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$J((W_3)_{[n-m]})(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large.}$$

Remark 5.20 If $J \in \text{Hom}(W_3, (W_1 \otimes W_2)^*)$ is \tilde{A} -compatible, then J is also grading restricted, as we see using (2.85).

Remark 5.21 Under the natural isomorphism given in Remark 5.18 (see (5.125)) in the \tilde{A} -compatible setting, the map $J : W'_3 \rightarrow (W_1 \otimes W_2)^*$ is grading restricted (recall Definition 5.19) if and only if the map $I : W_1 \otimes W_2 \rightarrow \overline{W_3}$ satisfies the lower truncation condition (4.3). But notice also that in this \tilde{A} -compatible setting, we have seen that both I and J automatically have these properties.

Using the above together with Remarks 5.4 and 5.12, we now have the following result, generalizing Proposition 13.1 in [HL7]:

Proposition 5.22 *Let W_1 , W_2 and W_3 be generalized V -modules. Under the natural isomorphism described in Remark 5.18 between the space of \tilde{A} -compatible linear maps*

$$I : W_1 \otimes W_2 \rightarrow \overline{W_3}$$

and the space of \tilde{A} -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

determined by (5.125), the $P(z)$ -intertwining maps I of type $\binom{W_3}{W_1 W_2}$ correspond exactly to the (grading restricted) \tilde{A} -compatible maps J that intertwine the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

and $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^$.*

Proof In view of (5.126), Remark 5.4 asserts that (5.81), or equivalently, (4.4), is equivalent to the condition

$$J \left(\tau_{W'_3} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) w'_{(3)} \right) = \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) J(w'_{(3)}), \quad (5.127)$$

that is, the condition that J intertwines the actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on W'_3 and on $(W_1 \otimes W_2)^*$ (recall (5.21)–(5.22)). Similarly, Remark 5.12 asserts that (4.5) is equivalent to the condition

$$J(L'(j)w'_{(3)}) = L'_{P(z)}(j)J(w'_{(3)}) \quad (5.128)$$

for $j = -1, 0, 1$, that is, the condition that J intertwines the actions of $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$. \square

Notation 5.23 Given generalized V -modules W_1 , W_2 and W_3 , we shall write $\mathcal{N}[P(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$, or $\mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*}$ if there is no ambiguity, for the space of (grading restricted) \tilde{A} -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

that intertwine the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

and $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$. Note that Proposition 5.22 gives a natural linear isomorphism

$$\begin{array}{ccc} \mathcal{M}[P(z)]_{W_1 W_2}^{W_3} = \mathcal{M}_{W_1 W_2}^{W_3} & \xrightarrow{\sim} & \mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*} \\ I & \mapsto & J \end{array}$$

(recall from Definition 4.2 the notations for the space of $P(z)$ -intertwining maps). Let us use the symbol “prime” to denote this isomorphism in both directions:

$$\begin{array}{ccc} \mathcal{M}_{W_1 W_2}^{W_3} & \xrightarrow{\sim} & \mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*} \\ I & \mapsto & I' \\ J' & \leftarrow & J, \end{array}$$

so that in particular,

$$I'' = I \quad \text{and} \quad J'' = J$$

for $I \in \mathcal{M}_{W_1 W_2}^{W_3}$ and $J \in \mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*}$, and the relation between I and I' is determined by

$$\langle w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle = I'(w'_{(3)})(w_{(1)} \otimes w_{(2)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, or equivalently,

$$w'_{(3)} \circ I = I'(w'_{(3)}).$$

Remark 5.24 Combining Proposition 5.22 with Proposition 4.7, we see that for any integer p , we also have a natural linear isomorphism

$$\mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*} \xrightarrow{\sim} \mathcal{V}_{W_1 W_2}^{W_3}$$

from $\mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*}$ to the space of logarithmic intertwining operators of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix} \right)$. In particular, given a logarithmic intertwining operator \mathcal{Y} of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix} \right)$, the map

$$I'_{\mathcal{Y}, p} : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

defined by

$$I'_{\mathcal{Y}, p}(w'_{(3)})(w_{(1)} \otimes w_{(2)}) = \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, e^{l_p(z)})w_{(2)} \rangle_{W_3}$$

is \tilde{A} -compatible and intertwines both actions on both spaces.

Recall that we have formulated the notions of $P(z)$ -product and $P(z)$ -tensor product using $P(z)$ -intertwining maps (Definitions 4.11 and 4.13). Since we now know that $P(z)$ -intertwining maps can be interpreted as in Proposition 5.22 (and Notation 5.23), we can easily reformulate the notions of $P(z)$ -product and $P(z)$ -tensor product correspondingly:

Proposition 5.25 *Let \mathcal{C}_1 be either of the categories \mathcal{M}_{sg} or \mathcal{GM}_{sg} , as in Definition 4.11. For $W_1, W_2 \in \text{ob } \mathcal{C}_1$, a $P(z)$ -product $(W_3; I_3)$ of W_1 and W_2 (recall Definition 4.11) amounts to an object (W_3, Y_3) of \mathcal{C}_1 equipped with a map $I'_3 \in \mathcal{N}_{W'_3}^{(W_1 \otimes W_2)^*}$, that is, equipped with an \tilde{A} -compatible map*

$$I'_3 : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

that intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ and of $\mathfrak{sl}(2)$. The map I'_3 corresponds to the $P(z)$ -intertwining map

$$I_3 : W_1 \otimes W_2 \rightarrow \overline{W_3}$$

as above:

$$I'_3(w'_{(3)}) = w'_{(3)} \circ I_3$$

for $w'_{(3)} \in W'_3$ (recall 5.126)). Denoting this structure by $(W_3, Y_3; I'_3)$ or simply by $(W_3; I'_3)$, let $(W_4; I'_4)$ be another such structure. Then a morphism of $P(z)$ -products from W_3 to W_4 amounts to a module map $\eta : W_3 \rightarrow W_4$ such that the diagram

$$\begin{array}{ccc} & (W_1 \otimes W_2)^* & \\ I'_4 \nearrow & & \nwarrow I'_3 \\ W'_4 & \xrightarrow{\eta'} & W'_3 \end{array}$$

commutes, where η' is the natural map given by (2.99).

Proof All we need to check is that the diagram in Definition 4.11 commutes if and only if the diagram above commutes. But this follows from the definitions and the fact that for $\overline{w_{(3)}} \in \overline{W_3}$ and $w'_{(4)} \in W'_4$,

$$\langle \eta'(w'_{(4)}), \overline{w_{(3)}} \rangle = \langle w'_{(4)}, \overline{\eta(w_{(3)})} \rangle,$$

which in turn follows from (2.99). \square

Corollary 5.26 *Let \mathcal{C} be a full subcategory of either \mathcal{M}_{sg} or \mathcal{GM}_{sg} , as in Definition 4.13. For $W_1, W_2 \in \text{ob } \mathcal{C}$, a $P(z)$ -tensor product $(W_0; I_0)$ of W_1 and W_2 in \mathcal{C} , if it exists, amounts to an object $W_0 = W_1 \boxtimes_{P(z)} W_2$ of \mathcal{C} and a structure $(W_0 = W_1 \boxtimes_{P(z)} W_2; I'_0)$ as in Proposition 5.25, with*

$$I'_0 : (W_1 \boxtimes_{P(z)} W_2)' \longrightarrow (W_1 \otimes W_2)^*$$

in $\mathcal{N}_{(W_1 \boxtimes_{P(z)} W_2)'}^{(W_1 \otimes W_2)^}$, such that for any such pair $(W; I')$ ($W \in \text{ob } \mathcal{C}$), with*

$$I' : W' \longrightarrow (W_1 \otimes W_2)^*$$

in $\mathcal{N}_{W'}^{(W_1 \otimes W_2)^*}$, there is a unique module map

$$\chi : W' \longrightarrow (W_1 \boxtimes_{P(z)} W_2)'$$

such that the diagram

$$\begin{array}{ccc} & (W_1 \otimes W_2)^* & \\ I' \nearrow & & \nwarrow I'_0 \\ W' & \xrightarrow{\chi} & (W_1 \boxtimes_{P(z)} W_2)' \end{array}$$

commutes. Here $\chi = \eta'$, where η is a correspondingly unique module map

$$\eta : W_1 \boxtimes_{P(z)} W_2 \longrightarrow W.$$

Also, the map I'_0 , which is \tilde{A} -compatible and which intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ and of $\mathfrak{sl}(2)$, is related to the $P(z)$ -intertwining map

$$I_0 = \boxtimes_{P(z)} : W_1 \otimes W_2 \longrightarrow \overline{W_1 \boxtimes_{P(z)} W_2}$$

by

$$I'_0(w') = w' \circ \boxtimes_{P(z)}$$

for $w' \in (W_1 \boxtimes_{P(z)} W_2)'$, that is,

$$I'_0(w')(w_{(1)} \otimes w_{(2)}) = \langle w', w_{(1)} \boxtimes_{P(z)} w_{(2)} \rangle$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, using the notation (4.30). \square

Remark 5.27 From Corollary 5.26 we see that it is natural to try to construct $W_1 \boxtimes_{P(z)} W_2$, when it exists, as the contragredient of a suitable natural substructure of $(W_1 \otimes W_2)^*$. We shall now proceed to do this. Under suitable assumptions, we shall in fact construct a module-like structure

$$W_1 \boxtimes_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$$

for $W_1, W_2 \in \text{ob } \mathcal{C}$, and we will show that $W_1 \boxtimes_{P(z)} W_2$ is an object of \mathcal{C} if and only if $W_1 \boxtimes_{P(z)} W_2$ exists in \mathcal{C} , in which case we will have

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxtimes_{P(z)} W_2)'$$

(observe the notation

$$\boxtimes = \boxtimes',$$

as in the special cases studied in [HL5]–[HL7]). It is important to keep in mind that the space $W_1 \boxtimes_{P(z)} W_2$ will depend on the category \mathcal{C} .

We formalize certain of the properties of the category \mathcal{C} that we have been using, and some new ones, as follows:

Assumption 5.28 *Throughout the remainder of this work, we shall assume that \mathcal{C} is a full subcategory of the category \mathcal{M}_{sg} or \mathcal{GM}_{sg} closed under the contragredient functor (recall Notation 2.36; for now, we are not assuming that $V \in \text{ob } \mathcal{C}$). We shall also assume that \mathcal{C} is closed under taking finite direct sums.*

Definition 5.29 For $W_1, W_2 \in \text{ob } \mathcal{C}$, define the subset

$$W_1 \boxtimes_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$$

of $(W_1 \otimes W_2)^*$ to be the union of the images

$$I'(W') \subset (W_1 \otimes W_2)^*$$

as $(W; I)$ ranges through all the $P(z)$ -products of W_1 and W_2 with $W \in \text{ob } \mathcal{C}$. Equivalently, $W_1 \boxtimes_{P(z)} W_2$ is the union of the images $I'(W')$ as W (or W') ranges through $\text{ob } \mathcal{C}$ and I' ranges through $\mathcal{N}_{W'}^{(W_1 \otimes W_2)^*}$ —the space of \tilde{A} -compatible linear maps

$$W' \rightarrow (W_1 \otimes W_2)^*$$

intertwining the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

and $\mathfrak{sl}(2)$ on both spaces.

Remark 5.30 Since \mathcal{C} is closed under direct sums (Assumption 5.28), it is clear that $W_1 \boxtimes_{P(z)} W_2$ is in fact a linear subspace of $(W_1 \otimes W_2)^*$, and in particular, it can be defined alternatively as the sum of all the images $I'(W')$:

$$W_1 \boxtimes_{P(z)} W_2 = \sum I'(W') = \bigcup I'(W') \subset (W_1 \otimes W_2)^*, \quad (5.129)$$

where the sum and union both range over $W \in \text{ob } \mathcal{C}$, $I \in \mathcal{M}_{W_1 W_2}^W$.

For any generalized V -modules W_1 and W_2 , using the operator $L'_{P(z)}(0)$ (recall (5.109)) on $(W_1 \otimes W_2)^*$ we define the generalized $L'_{P(z)}(0)$ -eigenspaces $((W_1 \otimes W_2)^*)_{[n]}$ for $n \in \mathbb{C}$ in the usual way:

$$((W_1 \otimes W_2)^*)_{[n]} = \{w \in (W_1 \otimes W_2)^* \mid (L'_{P(z)}(0) - n)^m w = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large}\}. \quad (5.130)$$

Then we have the (proper) subspace

$$\prod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{[n]} \subset (W_1 \otimes W_2)^*. \quad (5.131)$$

We also define the ordinary $L'_{P(z)}(0)$ -eigenspaces $((W_1 \otimes W_2)^*)_{(n)}$ in the usual way:

$$((W_1 \otimes W_2)^*)_{(n)} = \{w \in (W_1 \otimes W_2)^* \mid L'_{P(z)}(0)w = nw\}. \quad (5.132)$$

Then we have the (proper) subspace

$$\prod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{(n)} \subset (W_1 \otimes W_2)^*. \quad (5.133)$$

Proposition 5.31 *Let $W_1, W_2 \in \text{ob } \mathcal{C}$.*

(a) *The elements of $W_1 \boxtimes_{P(z)} W_2$ are exactly the linear functionals on $W_1 \otimes W_2$ of the form $w' \circ I(\cdot \otimes \cdot)$ for some $P(z)$ -intertwining map I of type $\binom{W}{W_1 W_2}$ and some $w' \in W'$, $W \in \text{ob } \mathcal{C}$.*

(b) *Let $(W; I)$ be any $P(z)$ -product of W_1 and W_2 , with W any generalized V -module. Then for $n \in \mathbb{C}$,*

$$I'(W'_{[n]}) \subset ((W_1 \otimes W_2)^*)_{[n]}$$

and

$$I'(W'_{(n)}) \subset ((W_1 \otimes W_2)^*)_{(n)}.$$

(c) *The structure $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ (recall (5.85)) satisfies all the axioms in the definition of (strongly \tilde{A} -graded) generalized V -module except perhaps for the two grading conditions (2.85) and (2.86).*

(d) *Suppose that the objects of the category \mathcal{C} consist only of (strongly \tilde{A} -graded) ordinary, as opposed to generalized, V -modules. Then the structure $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ satisfies all the axioms in the definition of (strongly \tilde{A} -graded ordinary) V -module except perhaps for (2.85) and (2.86).*

Proof Part (a) is clear from the definition of $W_1 \boxtimes_{P(z)} W_2$, and (b) follows from (5.128) with $j = 0$.

For (c), let $(W; I)$ be any any $P(z)$ -product of W_1 and W_2 , with W any generalized V -module. Then $(I'(W'), Y'_{P(z)})$ satisfies all the conditions in the definition of (strongly \tilde{A} -graded) generalized V -module since I' is \tilde{A} -compatible and intertwines the actions of $V \otimes \mathbb{C}[t, t^{-1}]$ and of $\mathfrak{sl}(2)$; the \mathbb{C} -grading follows from Part (b). Since $W_1 \boxtimes_{P(z)} W_2$ is the sum of these structures $I'(W')$ over $W \in \text{ob } \mathcal{C}$ (recall (5.129)), we see that $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ satisfies all the conditions in the definition of generalized module except perhaps for (2.85) and (2.86).

Finally, Part (d) is proved by the same argument as for (c). In fact, for $(W; I)$ any $P(z)$ -product of possibly generalized V -modules W_1 and W_2 , with W any ordinary V -module, $(I'(W'), Y'_{P(z)})$ satisfies all the conditions in the definition of (strongly \tilde{A} -graded) ordinary V -module; the \mathbb{C} -grading (this time, by ordinary $L'_{P(z)}(0)$ -eigenspaces) again follows from Part (b). \square

We now have the following generalization of Proposition 13.7 in [HL7], characterizing $W_1 \boxtimes_{P(z)} W_2$, including its existence, in terms of $W_1 \boxtimes_{P(z)} W_2$:

Proposition 5.32 *Let $W_1, W_2 \in \text{ob } \mathcal{C}$. If $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ is an object of \mathcal{C} , denote by $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ its contragredient module. Then the $P(z)$ -tensor product of W_1 and W_2 in \mathcal{C} exists and is $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; i')$, where i is the natural inclusion from $W_1 \boxtimes_{P(z)} W_2$ to $(W_1 \otimes W_2)^*$ (recall Notation 5.23). Conversely, let us assume that \mathcal{C} is closed under quotients. If the $P(z)$ -tensor product of W_1 and W_2 in \mathcal{C} exists, then $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ is an object of \mathcal{C} .*

Proof Suppose that $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$ is an object of \mathcal{C} and take $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ and the map i as indicated. Then

$$i \in \mathcal{N}_{W_1 \boxtimes_{P(z)} W_2}^{W_1 \otimes W_2^*},$$

and

$$i' \in \mathcal{M}_{W_1 W_2}^{W_1 \boxtimes_{P(z)} W_2}.$$

In the notation of Corollary 5.26, we take $I_0 = i'$, $I'_0 = i$. For any pair $(W; I')$ as in Corollary 5.26, we have $I'(W') \subset W_1 \boxtimes_{P(z)} W_2$ (which is the union of all such images), so that there is certainly a unique module map

$$\chi : W' \rightarrow W_1 \boxtimes_{P(z)} W_2$$

such that

$$i \circ \chi = I',$$

namely, I' itself, viewed as a module map. Thus by Corollary 5.26, $W_1 \boxtimes_{P(z)} W_2$ exists as indicated.

Conversely, if the $P(z)$ -tensor product of W_1 and W_2 in \mathcal{C} exists and is $(W_0; I_0)$, then for any $P(z)$ -product $(W; I)$ with $W \in \text{ob } \mathcal{C}$, we have a unique module map $\chi : W' \rightarrow W'_0$ as in Corollary 5.26 such that $I' = I'_0 \circ \chi$, so that $I'(W') \subset I'_0(W'_0)$, proving that $W_1 \boxtimes_{P(z)} W_2 \subset I'_0(W'_0)$. On the other hand, $(W_0; I_0)$ is itself a $P(z)$ -product, so that $I'_0(W'_0) \subset W_1 \boxtimes_{P(z)} W_2$. Thus $W_1 \boxtimes_{P(z)} W_2 = I'_0(W'_0)$, and so $W_1 \boxtimes_{P(z)} W_2$ is a generalized V -module and is the image of the module map

$$I'_0 : W'_0 \rightarrow W_1 \boxtimes_{P(z)} W_2.$$

Since \mathcal{C} is closed under quotients by assumption, we have that $W_1 \boxtimes_{P(z)} W_2 \in \text{ob } \mathcal{C}$. \square

Remark 5.33 Suppose that $W_1 \boxtimes_{P(z)} W_2$ is an object of \mathcal{C} . From Corollary 5.26 and Proposition 5.32 we see that

$$\langle \lambda, w_{(1)} \boxtimes_{P(z)} w_{(2)} \rangle \Big|_{W_1 \boxtimes_{P(z)} W_2} = \lambda(w_{(1)} \otimes w_{(2)}) \quad (5.134)$$

for $\lambda \in W_1 \boxtimes_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

Our next goal is to present a crucial alternative description of the subspace $W_1 \boxtimes_{P(z)} W_2$ of $(W_1 \otimes W_2)^*$. The main ingredient of this description will be the “ $P(z)$ -compatibility condition,” which was a cornerstone of the development of tensor product theory in the special cases treated in [HL5]–[HL7] and [H1].

Assume now that W_1 and W_2 are arbitrary generalized V -modules. Let $(W; I)$ (W a generalized V -module) be a $P(z)$ -product of W_1 and W_2 and let $w' \in W'$. Then from (5.127), Proposition 5.25, (5.124), (5.7) and (5.85), we have, for all $v \in V$,

$$\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) I'(w')$$

$$\begin{aligned}
&= I' \left(\tau_{W'} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) w' \right) \\
&= I' \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_{W'}(v, x_1) w' \right) \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) I'(Y_{W'}(v, x_1) w') \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) I'(\tau_{W'}(Y_t(v, x_1)) w') \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) \tau_{P(z)}(Y_t(v, x_1)) I'(w') \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1) I'(w'). \tag{5.135}
\end{aligned}$$

That is, $I'(w')$ satisfies the following nontrivial and subtle condition on $\lambda \in (W_1 \otimes W_2)^*$:

The $P(z)$ -compatibility condition

- (a) The $P(z)$ -lower truncation condition: For all $v \in V$, the formal Laurent series $Y'_{P(z)}(v, x)\lambda$ involves only finitely many negative powers of x .
- (b) The following formula holds:

$$\begin{aligned}
&\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1) \lambda \quad \text{for all } v \in V. \tag{5.136}
\end{aligned}$$

(Note that the two sides of (5.136) are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2)^*$. Note also that Condition (a) insures that the right-hand side in Condition (b) is well defined.)

Notation 5.34 Note that the set of elements of $(W_1 \otimes W_2)^*$ satisfying either the full $P(z)$ -compatibility condition or Part (a) of this condition forms a subspace. We shall denote the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition by

$$\text{COMP}_{P(z)}((W_1 \otimes W_2)^*).$$

Recall from the comments preceding Definition 5.5 that for each $\beta \in \tilde{A}$ we have the subspace $((W_1 \otimes W_2)^*)^{(\beta)}$ of $(W_1 \otimes W_2)^*$. The sum of these subspaces is of course direct:

$$\sum_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)^{(\beta)} = \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)^{(\beta)}.$$

Each space $((W_1 \otimes W_2)^*)^{(\beta)}$ is $L'_{P(z)}(0)$ -stable (recall Proposition 5.6 and Remark 5.13), so that we may consider the subspaces

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{[n]}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

and

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{(n)}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

(recall Remark 2.13). We now define the two subspaces

$$((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})} = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{[n]}^{(\beta)} \subset (W_1 \otimes W_2)^* \quad (5.137)$$

and

$$((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})} = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{(n)}^{(\beta)} \subset (W_1 \otimes W_2)^*. \quad (5.138)$$

Remark 5.35 Any $L'_{P(z)}(0)$ -stable subspace of $((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}$ is graded by generalized eigenspaces (again recall Remark 2.13), and if such a subspace is also \tilde{A} -graded, then it is doubly graded; similarly for subspaces of $((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}$.

We have:

Lemma 5.36 *Suppose that $\lambda \in ((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}$ satisfies the $P(z)$ -compatibility condition. Then every \tilde{A} -homogeneous component of λ also satisfies this condition.*

Proof When $v \in V$ is \tilde{A} -homogeneous,

$$\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \quad \text{and} \quad x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1)$$

are both \tilde{A} -homogeneous as operators, in the obvious sense. By comparing the \tilde{A} -homogeneous components of both sides of (5.136), we see that the \tilde{A} -homogeneous components of λ also satisfy the $P(z)$ -compatibility condition. \square

Remark 5.37 Both the spaces $((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}$ and $((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}$ are stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. For the \tilde{A} -grading, this follows from Proposition 5.6 and Remark 5.13, and for the \mathbb{C} -gradings, we simply follow the proof of Proposition 2.19, using Propositions 5.8 and 5.9 together with (5.114).

Again let $(W; I)$ (W a generalized V -module) be a $P(z)$ -product of W_1 and W_2 and let $w' \in W'$. Since I' in particular intertwines the actions of $V \otimes \mathbb{C}[t, t^{-1}]$ and of $\mathfrak{sl}(2)$, and is \tilde{A} -compatible, $I'(W')$ is a generalized V -module, as we have seen in the proof of Proposition 5.31. Therefore, for each $w' \in W'$, $I'(w')$ also satisfies the following condition on $\lambda \in (W_1 \otimes W_2)^*$:

The $P(z)$ -local grading restriction condition

(a) The $P(z)$ -grading condition: λ is a (finite) sum of generalized eigenvectors for the operator $L'_{P(z)}(0)$ on $(W_1 \otimes W_2)^*$ that are also homogeneous with respect to \tilde{A} , that is,

$$\lambda \in ((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}.$$

(b) Let W_λ be the smallest doubly graded (or equivalently, \tilde{A} -graded; recall Remark 5.35) subspace of $((W_1 \otimes W_2)^*)_{[\mathbb{C}]}^{(\tilde{A})}$ containing λ and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. (In view of Remark 5.37, W_λ indeed exists.) Then W_λ has the properties

$$\dim(W_\lambda)_{[n]}^{(\beta)} < \infty, \tag{5.139}$$

$$(W_\lambda)_{[n+k]}^{(\beta)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative,} \tag{5.140}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where as usual the subscripts denote the \mathbb{C} -grading and the superscripts denote the \tilde{A} -grading.

In the case that W is an (ordinary) V -module and $w' \in W'$, $I'(w')$ also satisfies the following $L(0)$ -semisimple version of this condition on $\lambda \in (W_1 \otimes W_2)^*$:

The $L(0)$ -semisimple $P(z)$ -local grading restriction condition

(a) The $L(0)$ -semisimple $P(z)$ -grading condition: λ is a (finite) sum of eigenvectors for the operator $L'_{P(z)}(0)$ on $(W_1 \otimes W_2)^*$ that are also homogeneous with respect to \tilde{A} , that is,

$$\lambda \in ((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}.$$

(b) Consider W_λ as above, which in this case is in fact the smallest doubly graded (or equivalently, \tilde{A} -graded) subspace of $((W_1 \otimes W_2)^*)_{(\mathbb{C})}^{(\tilde{A})}$ containing λ and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. Then W_λ has the properties

$$\dim(W_\lambda)_{(n)}^{(\beta)} < \infty, \tag{5.141}$$

$$(W_\lambda)_{(n+k)}^{(\beta)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative,} \tag{5.142}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the \mathbb{C} -grading and the superscripts denote the \tilde{A} -grading.

Notation 5.38 Note that the set of elements of $(W_1 \otimes W_2)^*$ satisfying either of these two $P(z)$ -local grading restriction conditions, or either of the Part (a)'s in these conditions, forms a subspace. We shall denote the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -local grading restriction condition and the $L(0)$ -semisimple $P(z)$ -local grading restriction condition by

$$\text{LGR}_{[\mathbb{C}];P(z)}((W_1 \otimes W_2)^*)$$

and

$$\text{LGR}_{(\mathbb{C});P(z)}((W_1 \otimes W_2)^*),$$

respectively.

The following theorems are among the most important in this work. Note that even in the finitely reductive case studied in [HL7], they are stronger and more general than (the last assertion of) Theorem 13.9 in [HL7]. The proofs of these two theorems will be given in the next section.

Theorem 5.39 *Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition. Then when acting on λ , the Jacobi identity for $Y'_{P(z)}$ holds, that is,*

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2) \lambda \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{P(z)}(v, x_2) Y'_{P(z)}(u, x_1) \lambda \\ & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{P(z)}(Y(u, x_0)v, x_2) \lambda \end{aligned} \tag{5.143}$$

for $u, v \in V$.

Theorem 5.40 *The subspace $\text{COMP}_{P(z)}((W_1 \otimes W_2)^*)$ of $(W_1 \otimes W_2)^*$ is stable under the operators $\tau_{P(z)}(v \otimes t^n)$ for $v \in V$ and $n \in \mathbb{Z}$, and in the Möbius case, also under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$; similarly for the subspaces $\text{LGR}_{[\mathbb{C}];P(z)}((W_1 \otimes W_2)^*$ and $\text{LGR}_{(\mathbb{C});P(z)}((W_1 \otimes W_2)^*$.*

Remark 5.41 The converse of Theorem 5.39 is not true. One can see this in the tensor product theory of the “trivial” case where V is a vertex operator algebra associated with a finite-dimensional unital commutative associative algebra $(A, \cdot, 1)$ with derivation $D = 0$ (cf. Remark 2.3). In this case, $(V, Y, \mathbf{1}, \omega) = (A, \cdot, 1, 0)$ and the Jacobi identity for a V -module W reduces to

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) u \cdot (v \cdot w) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) v \cdot (u \cdot w) \\ & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (u \cdot v) \cdot w \end{aligned}$$

for $u, v \in A$ and $w \in W$, where we also use “.” to denote the action of A on its modules. In particular, a V -module is just a finite-dimensional module for the associative algebra A . Given V -modules W_1 and W_2 , the action $Y'_{P(z)}$ given in (5.87) now becomes

$$(Y'_{P(z)}(v, x)\lambda)(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes v \cdot w_{(2)}).$$

From this it is clear that (5.143) holds for any element $\lambda \in (W_1 \otimes W_2)^*$. However, the $P(z)$ -compatibility condition (5.136) in this case reduces to

$$\lambda(v \cdot w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes v \cdot w_{(2)})$$

for all $v \in A$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, which is not necessarily true for every λ . This example is discussed further in Remark 2.20 of [HLLZ], which treats a range of issues related to the compatibility condition, intertwining operators, and tensor product theory.

We now generalize the notion of “weak module” for a vertex operator algebra to our Möbius or conformal vertex algebra V :

Definition 5.42 A *weak module* for V (or *weak V -module*) is a vector space W equipped with a vertex operator map $Y_W : V \otimes W \rightarrow W[[x, x^{-1}]]$ satisfying (only) the axioms (2.35), (2.36), (2.37) and (2.40) in Definition 2.9 (note that there is no grading given on W) and in case V is Möbius, also the existence of a representation of $\mathfrak{sl}(2)$ on W , as in Definition 2.11, satisfying the conditions (2.28)–(2.30).

Then we have:

Theorem 5.43 *The space $\text{COMP}_{P(z)}((W_1 \otimes W_2)^*)$, equipped with the vertex operator map $Y'_{P(z)}$ and, in case V is Möbius, also equipped with the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$, is a weak V -module; similarly for the spaces*

$$(\text{COMP}_{P(z)}((W_1 \otimes W_2)^*)) \cap (\text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*)) \quad (5.144)$$

and

$$(\text{COMP}_{P(z)}((W_1 \otimes W_2)^*)) \cap (\text{LGR}_{(\mathbb{C}); P(z)}((W_1 \otimes W_2)^*)). \quad (5.145)$$

Proof By Theorem 5.40, $Y'_{P(z)}$ is a map from the tensor product of V with any of these three subspaces to the space of formal Laurent series with elements of the subspace as coefficients. By Proposition 5.8 and Theorem 5.39 and, in the case that V is a Möbius vertex algebra, also by Propositions 5.14 and 5.15, we see that all the axioms for weak V -module are satisfied. \square

We also have:

Theorem 5.44 *Let*

$$\lambda \in \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*).$$

Then W_λ (recall Part (b) of the $P(z)$ -local grading restriction condition) equipped with the vertex operator map $Y'_{P(z)}$ and, in case V is Möbius, also equipped with the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$, is a (strongly-graded) generalized V -module. If in addition

$$\lambda \in \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}); P(z)}((W_1 \otimes W_2)^*),$$

that is, λ is a sum of eigenvectors of $L'_{P(z)}(0)$, then $W_\lambda \subset ((W_1 \otimes W_2)^)_{(\mathbb{C})}^{(\tilde{A})}$ is a (strongly-graded) V -module.*

Proof Decompose λ as

$$\lambda = \sum_{\beta \in \tilde{A}} \lambda^{(\beta)}$$

(finite sum), where $\lambda^{(\beta)} \in ((W_1 \otimes W_2)^*)^{(\beta)}$. By Lemma 5.36, each $\lambda^{(\beta)}$ satisfies the $P(z)$ -compatibility condition. Also, each $\lambda^{(\beta)}$ satisfies the $P(z)$ -grading condition (and in the semisimple case, the $L(0)$ -semisimple $P(z)$ -grading condition), and each $W_{\lambda^{(\beta)}}$ is simply the smallest subspace containing $\lambda^{(\beta)}$ and stable under the operators listed above (without the \tilde{A} -gradedness condition). Moreover, each $W_{\lambda^{(\beta)}} \subset W_\lambda$ and in fact

$$W_\lambda = \sum_{\beta \in \tilde{A}} W_{\lambda^{(\beta)}}.$$

Thus each $\lambda^{(\beta)}$ lies in the space (5.144) (or (5.145)). (Note that we have reduced Theorem 5.44 to the \tilde{A} -homogeneous case.) By Theorem 5.43, each $W_{\lambda^{(\beta)}}$ is a weak submodule of the weak module (5.144) (or (5.145)), and hence is a (strongly-graded) generalized module (or module). Thus W_λ has the same properties. \square

Now we can give an alternative description of $W_1 \boxtimes_{P(z)} W_2$ by characterizing the elements of $W_1 \boxtimes_{P(z)} W_2$ using the $P(z)$ -compatibility condition and the $P(z)$ -local grading restriction conditions, generalizing Theorem 13.10 in [HL7]. This description will be crucial in later sections, especially in the construction of the associativity isomorphisms.

Theorem 5.45 *Suppose that for every element*

$$\lambda \in \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*)$$

the (strongly-graded) generalized module W_λ given in Theorem 5.44 is an object of \mathcal{C} (this of course holds in particular if \mathcal{C} is \mathcal{GM}_{sg}). Then

$$W_1 \boxtimes_{P(z)} W_2 = \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; P(z)}((W_1 \otimes W_2)^*).$$

Suppose that \mathcal{C} is a category of strongly-graded V -modules (that is, $\mathcal{C} \subset \mathcal{M}_{sg}$) and that for every element

$$\lambda \in \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}; P(z))}((W_1 \otimes W_2)^*)$$

the (strongly-graded) V -module W_λ given in Theorem 5.44 is an object of \mathcal{C} (which of course holds in particular if \mathcal{C} is \mathcal{M}_{sg}). Then

$$W_1 \boxtimes_{P(z)} W_2 = \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}; P(z))}((W_1 \otimes W_2)^*).$$

Proof We have seen that

$$W_1 \boxtimes_{P(z)} W_2 \subset \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}; P(z)]}((W_1 \otimes W_2)^*)$$

and, in case $\mathcal{C} \subset \mathcal{M}_{sg}$,

$$W_1 \boxtimes_{P(z)} W_2 \subset \text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}; P(z))}((W_1 \otimes W_2)^*).$$

On the other hand, by the assumptions, every element λ of

$$\text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}; P(z)]}((W_1 \otimes W_2)^*)$$

and, in case $\mathcal{C} \subset \mathcal{M}_{sg}$, every element λ of

$$\text{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}; P(z))}((W_1 \otimes W_2)^*),$$

is contained in some object of \mathcal{C} , namely, W_λ , and for any such (generalized) module, the inclusion map into $(W_1 \otimes W_2)^*$ satisfies the intertwining conditions in Proposition 5.22. Thus λ lies in $W_1 \boxtimes_{P(z)} W_2$, proving the desired inclusion. \square

5.3 Constructions of $Q(z)$ -tensor products

We now give the construction of $Q(z)$ -tensor products. It is analogous to that of $P(z)$ -tensor products, and the formulations, results and proofs in this subsection very closely parallel those in Subsection 5.2. As usual, $z \in \mathbb{C}^\times$.

Given generalized V -modules W_1 and W_2 , we shall be constructing an action of the space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on the space $(W_1 \otimes W_2)^*$.

Let I be a $Q(z)$ -intertwining map of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$, as in Definition 4.32. Consider the contragredient generalized V -module (W'_3, Y'_3) , recall the opposite vertex operator (2.57) and formula (2.73), and recall why the ingredients of formula (4.72) are well defined. For $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, applying $w'_{(3)}$ to (4.72) we obtain

$$\begin{aligned} & \left\langle z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y'_3(v, x_0) w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \right\rangle \\ &= \left\langle w'_{(3)}, x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) I(Y_1^o(v, x_1) w_{(1)} \otimes w_{(2)}) \right\rangle \\ & \quad - \left\langle w'_{(3)}, x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) I(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \right\rangle. \end{aligned} \tag{5.146}$$

We shall use this to motivate our action.

As we discussed in Subsection 5.1 (see (5.12) and (5.13)), in the left-hand side of (5.146), the coefficients of

$$z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right)Y'_3(v, x_0) \quad (5.147)$$

in powers of x_0 and x_1 , for all $v \in V$, span

$$\tau_{W'_3}(V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]) \quad (5.148)$$

(recall (5.2) and (5.7)). We now define a linear action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on $(W_1 \otimes W_2)^*$, that is, a linear map

$$\tau_{Q(z)} : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \rightarrow \text{End}(W_1 \otimes W_2)^*.$$

Recall the notations T_{-z}^+ and T_{-z}^o from Subsection 5.1 ((5.72) and (5.75)).

Definition 5.46 We define the linear action $\tau_{Q(z)}$ of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$$

on $(W_1 \otimes W_2)^*$ by

$$(\tau_{Q(z)}(\xi)\lambda)(w_{(1)} \otimes w_{(2)}) = \lambda(\tau_{W_1}(T_{-z}^o \xi)w_{(1)} \otimes w_{(2)}) - \lambda(w_{(1)} \otimes \tau_{W_2}(T_{-z}^+ \xi)w_{(2)}) \quad (5.149)$$

for $\xi \in V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and denote by $Y'_{Q(z)}$ the action of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2)^*$ thus defined, that is,

$$Y'_{Q(z)}(v, x) = \tau_{Q(z)}(Y_t(v, x)) \quad (5.150)$$

for $v \in V$.

Using Lemma 5.2, (5.7) and (5.61), we see that (5.149) can be written using generating functions as

$$\begin{aligned} & \left(\tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \lambda(Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \end{aligned} \quad (5.151)$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$; compare this with (5.146). The generating function form of the action $Y'_{Q(z)}$ can be obtained by taking Res_{x_1} of both sides of (5.151):

$$(Y'_{Q(z)}(v, x_0)\lambda)(w_{(1)} \otimes w_{(2)})$$

$$\begin{aligned}
&= \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \lambda(Y_1^o(v, x_1) w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \\
&= \lambda(Y_1^o(v, x_0 + z) w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{x_1} x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}). \tag{5.152}
\end{aligned}$$

Remark 5.47 Using the actions $\tau_{W'_3}$ and $\tau_{Q(z)}$, we can write (5.146) as

$$\left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_3(v, x_0) w'_{(3)} \right) \circ I = \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) (w'_{(3)} \circ I)$$

or equivalently, as

$$\left(\tau_{W'_3} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) w'_{(3)} \right) \circ I = \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) (w'_{(3)} \circ I).$$

Recall the \tilde{A} -grading on $(W_1 \otimes W_2)^*$ and the A -grading on $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$. Similarly, we also have an A -grading on $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$. Definition 5.5 also applies to a linear action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$ on $(W_1 \otimes W_2)^*$. From (5.149) or (5.151), we have:

Proposition 5.48 *The action $\tau_{Q(z)}$ is \tilde{A} -compatible.* \square

We also have:

Proposition 5.49 *The action $Y'_{Q(z)}$ has the property*

$$Y'_{Q(z)}(\mathbf{1}, x) = 1 \tag{5.153}$$

and the $L(-1)$ -derivative property

$$\frac{d}{dx} Y'_{Q(z)}(v, x) = Y'_{Q(z)}(L(-1)v, x) \tag{5.154}$$

for $v \in V$.

Proof From (5.152), (2.57) and (2.7),

$$\begin{aligned}
(Y'_{Q(z)}(\mathbf{1}, x) \lambda)(w_{(1)} \otimes w_{(2)}) &= \text{Res}_{x_1} x^{-1} \delta \left(\frac{x_1 - z}{x} \right) \lambda(w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{x_1} x^{-1} \delta \left(\frac{z - x_1}{-x} \right) \lambda(w_{(1)} \otimes w_{(2)}) \\
&= \text{Res}_{x_1} x_1^{-1} \delta \left(\frac{z + x}{x_1} \right) \lambda(w_{(1)} \otimes w_{(2)}) \\
&= \lambda(w_{(1)} \otimes w_{(2)}), \tag{5.155}
\end{aligned}$$

proving (5.153). To prove the $L(-1)$ -derivative property, observe that from (5.152),

$$\begin{aligned} & \left(\left(\frac{d}{dx} Y'_{Q(z)}(v, x) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= \frac{d}{dx} \lambda(Y_1^o(v, x+z) w_{(1)} \otimes w_{(2)}) \\ &+ \text{Res}_{x_1} \left(\frac{d}{dx} z^{-1} \delta \left(\frac{-x+x_1}{z} \right) \right) \lambda(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}). \end{aligned} \quad (5.156)$$

But for any formal Laurent series $f(x)$, we have

$$\frac{d}{dx} f \left(\frac{-x+x_1}{z} \right) = -\frac{d}{dx_1} f \left(\frac{-x+x_1}{z} \right) \quad (5.157)$$

and if $f(x)$ involves only finitely many negative powers of x ,

$$\text{Res}_{x_1} \left(\frac{d}{dx_1} z^{-1} \delta \left(\frac{-x+x_1}{z} \right) \right) f(x_1) = -\text{Res}_{x_1} z^{-1} \delta \left(\frac{-x+x_1}{z} \right) \frac{d}{dx_1} f(x_1) \quad (5.158)$$

(since the residue of a derivative is 0). We also have the $L(-1)$ -derivative property (2.62) for Y^o . Thus the right-hand side of (5.156) equals

$$\begin{aligned} & \lambda(Y_1^o(L(-1)v, x+z) w_{(1)} \otimes w_{(2)}) \\ &+ \text{Res}_{x_1} z^{-1} \delta \left(\frac{-x+x_1}{z} \right) \frac{d}{dx_1} \lambda(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \\ &= \lambda(Y_1^o(L(-1)v, x+z) w_{(1)} \otimes w_{(2)}) \\ &+ \text{Res}_{x_1} z^{-1} \delta \left(\frac{-x+x_1}{z} \right) \lambda(w_{(1)} \otimes Y_2(L(-1)v, x_1) w_{(2)}) \\ &= (Y'_{Q(z)}(L(-1)v, x) \lambda)(w_{(1)} \otimes w_{(2)}), \end{aligned} \quad (5.159)$$

proving (5.154). \square

Proposition 5.50 *The action $Y'_{Q(z)}$ satisfies the commutator formula for vertex operators: On $(W_1 \otimes W_2)^*$,*

$$\begin{aligned} & [Y'_{Q(z)}(v_1, x_1), Y'_{Q(z)}(v_2, x_2)] \\ &= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1-x_0}{x_2} \right) Y'_{Q(z)}(Y(v_1, x_0) v_2, x_2) \end{aligned} \quad (5.160)$$

for $v_1, v_2 \in V$.

Proof As usual, the reader should note the well-definedness of each expression and the justifiability of each use of a δ -function property in the argument that follows. This argument

is the same as the proof of Proposition 5.2 of [HL5], given in Section 8 of [HL6]. Let $\lambda \in (W_1 \otimes W_2)^*$, $v_1, v_2 \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. By (5.152),

$$\begin{aligned}
& (Y'_{Q(z)}(v_1, x_1)Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(Y_1^o(v_1, y_1)w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&= \text{Res}_{y_1} \text{Res}_{y_2} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_2, y_2)Y_1^o(v_1, y_1)w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{y_1} \text{Res}_{y_2} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_1, y_1)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
&\quad - \text{Res}_{y_1} \text{Res}_{y_2} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_2, y_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&\quad + \text{Res}_{y_1} \text{Res}_{y_2} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}). \tag{5.161}
\end{aligned}$$

Transposing the subscripts 1 and 2 of the symbols v , x and y , we have

$$\begin{aligned}
& (Y'_{Q(z)}(v_2, x_2)Y'_{Q(z)}(v_1, x_1)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_1, y_1)Y_1^o(v_2, y_2)w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_2, y_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&\quad - \text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_1, y_1)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
&\quad + \text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_1, y_1)Y_2(v_2, y_2)w_{(2)}). \tag{5.162}
\end{aligned}$$

Formulas (5.161) and (5.162) give

$$([Y'_{Q(z)}(v_1, x_1), Y'_{Q(z)}(v_2, x_2)]\lambda)(w_{(1)} \otimes w_{(2)})$$

$$\begin{aligned}
&= \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \cdot \\
&\quad \cdot \lambda([Y_1^o(v_2, y_2), Y_1^o(v_1, y_1)]w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes [Y_2(v_1, y_1), Y_2(v_2, y_2)]w_{(2)}) \\
&= \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \cdot \\
&\quad \cdot \lambda \left(\text{Res}_{x_0} y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) Y_1^o(Y(v_1, x_0)v_2, y_2)w_{(1)} \otimes w_{(2)} \right) \\
&\quad - \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
&\quad \cdot \lambda \left(w_{(1)} \otimes \text{Res}_{x_0} y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) Y_2(Y(v_1, x_0)v_2, y_2)w_{(2)} \right) \\
&= \text{Res}_{x_0} \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(Y(v_1, x_0)v_2, y_2)w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{x_0} \text{Res}_{y_2} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(Y(v_1, x_0)v_2, y_2)w_{(2)}). \tag{5.163}
\end{aligned}$$

But

$$\begin{aligned}
&x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= y_1^{-1} \delta \left(\frac{x_1 + z}{y_1} \right) y_2^{-1} \delta \left(\frac{x_2 + z}{y_2} \right) (x_2 + z)^{-1} \delta \left(\frac{(x_1 + z) - x_0}{x_2 + z} \right) \\
&= y_1^{-1} \delta \left(\frac{x_1 + z}{y_1} \right) y_2^{-1} \delta \left(\frac{x_2 + z}{y_2} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \\
&= y_1^{-1} \delta \left(\frac{x_1 + z}{y_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \tag{5.164}
\end{aligned}$$

and

$$\begin{aligned}
&x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= z^{-1} \delta \left(\frac{-x_1 + y_1}{z} \right) z^{-1} \delta \left(\frac{-x_2 + y_2}{z} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= \left(\sum_{m, n \in \mathbb{Z}} \frac{(-x_1 + y_1)^m}{z^{m+1}} \frac{(-x_2 + y_2)^n}{z^{n+1}} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{m,n \in \mathbb{Z}} (-x_2 + y_2)^{-1} \left(\frac{-x_1 + y_1}{-x_2 + y_2} \right)^m \frac{(-x_2 + y_2)^{m+n+1}}{z^{m+n+2}} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= \left(\sum_{m,k \in \mathbb{Z}} (-x_2 + y_2)^{-1} \left(\frac{-x_1 + y_1}{-x_2 + y_2} \right)^m z^{-1} \left(\frac{-x_2 + y_2}{z} \right)^k \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= (-x_2 + y_2)^{-1} \delta \left(\frac{-x_1 + y_1}{-x_2 + y_2} \right) z^{-1} \delta \left(\frac{-x_2 + y_2}{z} \right) y_2^{-1} \delta \left(\frac{y_1 - x_0}{y_2} \right) \\
&= (-x_2)^{-1} \delta \left(\frac{x_1 - (y_1 - y_2)}{x_2} \right) z^{-1} \delta \left(\frac{-x_2 + y_2}{z} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right) \\
&= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right). \tag{5.165}
\end{aligned}$$

Thus (5.163) becomes

$$\begin{aligned}
&([Y'_{Q(z)}(v_1, x_1), Y'_{Q(z)}(v_2, x_2)]\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \text{Res}_{x_0} \text{Res}_{y_2} \text{Res}_{y_1} y_1^{-1} \delta \left(\frac{x_1 + z}{y_1} \right) x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(Y(v_1, x_0)v_2, y_2)w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{x_0} \text{Res}_{y_2} \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) y_1^{-1} \delta \left(\frac{y_2 + x_0}{y_1} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(Y(v_1, x_0)v_2, y_2)w_{(2)}) \\
&= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \cdot \\
&\quad \cdot \left(\text{Res}_{y_2} x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \lambda(Y_1^o(Y(v_1, x_0)v_2, y_2)w_{(1)} \otimes w_{(2)}) \right. \\
&\quad \left. - \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(w_{(1)} \otimes Y_2(Y(v_1, x_0)v_2, y_2)w_{(2)}) \right) \\
&= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (Y'_{Q(z)}(Y(v_1, x_0)v_2, x_2)\lambda)(w_{(1)} \otimes w_{(2)}). \tag{5.166}
\end{aligned}$$

Since λ , $w_{(1)}$ and $w_{(2)}$ are arbitrary, this equality gives the commutator formula (5.160) for $Y'_{Q(z)}$. \square

When V is in fact a conformal vertex algebra, we write

$$Y'_{Q(z)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{Q(z)}(n) x^{-n-2}. \tag{5.167}$$

Then from the last two propositions we see that the coefficient operators of $Y'_{Q(z)}(\omega, x)$ satisfy the Virasoro algebra commutator relations:

$$[L'_{Q(z)}(m), L'_{Q(z)}(n)] = (m - n)L'_{Q(z)}(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c. \tag{5.168}$$

Moreover, in this case, by setting $v = \omega$ in (5.152) and taking $\text{Res}_{x_0} x_0^{j+1}$ for $j = -1, 0, 1$, we see that

$$\begin{aligned}
& (L'_{Q(z)}(j)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&= \text{Res}_{x_1} (x_1 - z)^{j+1} \lambda(Y_1^o(\omega, x_1)w_{(1)} \otimes w_{(2)}) \\
&\quad - \text{Res}_{x_1} (-z + x_1)^{j+1} \lambda(w_{(1)} \otimes Y_2(\omega, x_1)w_{(2)}) \\
&= \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \lambda(L(i-j)w_{(1)} \otimes w_{(2)}) \\
&\quad - \sum_{i=0}^{j+1} \binom{j+1}{i} (-z)^i \lambda(w_{(1)} \otimes L(j-i)w_{(2)})
\end{aligned} \tag{5.169}$$

for $j = -1, 0, 1$. If V is just a Möbius vertex algebra, we define the actions $L'_{Q(z)}(j)$ on $(W_1 \otimes W_2)^*$ by the right-hand side of (5.169) for $j = -1, 0$ and 1 .

Remark 5.51 In view of the action $L'_{Q(z)}(j)$, the $\mathfrak{sl}(2)$ -bracket relations (4.73) for a $Q(z)$ -intertwining map can be written as

$$(L'(j)w'_{(3)}) \circ I = L'_{Q(z)}(j)(w'_{(3)} \circ I) \tag{5.170}$$

for $w'_{(3)} \in W'_3$ and $j = -1, 0$, and 1 .

Remark 5.52 We have

$$L'_{Q(z)}(j)((W_1 \otimes W_2)^*)^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

for $j = -1, 0, 1$ and $\beta \in \tilde{A}$ (cf. Proposition 5.48).

In the case that V is a conformal vertex algebra, $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$ realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$ (cf. (2.27)) on $(W_1 \otimes W_2)^*$. In the case that V is just a Möbius vertex algebra, we now state this fact as a proposition. This proposition is needed in the proof of Theorem 5.72 and therefore also for Theorems 5.73 and 5.74, but neither this proposition nor any of these three theorems are needed anywhere else in this work, so we omit the proof of this proposition. Of course, however, the proof is straightforward, as is the case with all the $\mathfrak{sl}(2)$ formulas.

Proposition 5.53 *Let V be a Möbius vertex algebra and let W_1 and W_2 be generalized V -modules. Then the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$ realize the actions of L_{-1} , L_0 and L_1 in $\mathfrak{sl}(2)$ on $(W_1 \otimes W_2)^*$.*

We also have:

Proposition 5.54 *Let V be a Möbius vertex algebra and let W_1 and W_2 be generalized V -modules. Then for $v \in V$,*

$$[L(-1), Y'_{Q(z)}(v, x)] = Y'_{Q(z)}(L(-1)v, x), \quad (5.171)$$

$$[L(0), Y'_{Q(z)}(v, x)] = Y'_{Q(z)}(L(0)v, x) + xY'_{Q(z)}(L(-1)v, x), \quad (5.172)$$

$$[L(1), Y'_{Q(z)}(v, x)] = Y'_{Q(z)}(L(1)v, x) + 2xY'_{Q(z)}(L(0)v, x) + x^2Y'_{Q(z)}(L(-1)v, x), \quad (5.173)$$

where for brevity we write $L'_{Q(z)}(j)$ acting on $(W_1 \otimes W_2)^*$ as $L(j)$.

Proof We prove only (5.172) since it is needed for Remark 5.68 and in Section 6. We omit the proofs of (5.171) and (5.173) for the same reasons as above; they are used only for Theorems 5.72–5.74.

Let $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Using (5.169), (5.152), the commutator formulas for $L(j)$ and $Y_1(v, x_0)$ for $j = -1, 0, 1$ and $v \in V$ (recall Definition 2.11), and the commutator formulas for $L(j)$ and $Y_2^o(v, x)$ for $j = -1, 0, 1$ and $v \in V$ (recall Lemma 2.22), we obtain

$$\begin{aligned} & ([L(0), Y'_{Q(z)}(v, x)]\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= (Y'_{Q(z)}(v, x)\lambda)(L(0)w_{(1)} \otimes w_{(2)}) \\ &\quad - z(Y'_{Q(z)}(v, x)\lambda)(L(1)w_{(1)} \otimes w_{(2)}) \\ &\quad - (Y'_{Q(z)}(v, x)\lambda)(w_{(1)} \otimes L(0)w_{(2)}) \\ &\quad + z(Y'_{Q(z)}(v, x)\lambda)(w_{(1)} \otimes L(-1)w_{(2)}) \\ &\quad - (L(0)\lambda)(Y_1^o(v, x+z)w_{(1)} \otimes w_{(2)}) \\ &\quad + \text{Res}_{x_1} x^{-1} \delta \left(\frac{z-x_1}{-x} \right) (L(0)\lambda)(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\ &= \lambda(Y_1^o(v, x+z)L(0)w_{(1)} \otimes w_{(2)}) \\ &\quad - \text{Res}_{x_1} x^{-1} \delta \left(\frac{z-x_1}{-x} \right) \lambda(L(0)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\ &\quad - z\lambda(Y_1^o(v, x+z)L(1)w_{(1)} \otimes w_{(2)}) \\ &\quad + z\text{Res}_{x_1} x^{-1} \delta \left(\frac{z-x_1}{-x} \right) \lambda(L(1)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\ &\quad - \lambda(Y_1^o(v, x+z)w_{(1)} \otimes L(0)w_{(2)}) \\ &\quad + \text{Res}_{x_1} x^{-1} \delta \left(\frac{z-x_1}{-x} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1)L(0)w_{(2)}) \\ &\quad + z\lambda(Y_1^o(v, x+z)w_{(1)} \otimes L(-1)w_{(2)}) \\ &\quad - z\text{Res}_{x_1} x^{-1} \delta \left(\frac{z-x_1}{-x} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1)L(-1)w_{(2)}) \\ &\quad - \lambda(L(0)Y_1^o(v, x+z)w_{(1)} \otimes w_{(2)}) \\ &\quad + z\lambda(L(1)Y_1^o(v, x+z)w_{(1)} \otimes w_{(2)}) \end{aligned}$$

$$\begin{aligned}
& +\lambda(Y_1^o(v, x+z)w_{(1)} \otimes L(0)w_{(2)}) \\
& -z\lambda(Y_1^o(v, x+z)w_{(1)} \otimes L(-1)w_{(2)}) \\
& +\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(L(0)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\
& -z\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(L(1)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes L(0)Y_2(v, x_1)w_{(2)}) \\
& +z\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes L(-1)Y_2(v, x_1)w_{(2)}) \\
& =\lambda([Y_1^o(v, x+z), L(0)]w_{(1)} \otimes w_{(2)}) \\
& -z\lambda([Y_1^o(v, x+z), L(1)]w_{(1)} \otimes w_{(2)}) \\
& +z\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes [L(-1), Y_2(v, x_1)]w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes [L(0), Y_2(v, x_1)]w_{(2)}) \\
& =\lambda(Y_1^o((L(0) + (x+z)L(-1))v, x+z)w_{(1)} \otimes w_{(2)}) \\
& -z\lambda(Y_1^o(L(-1)v, x+z)w_{(1)} \otimes w_{(2)}) \\
& +z\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes Y_2(L(-1)v, x_1)w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes Y_2((L(0) + x_1L(-1))v, x_1)w_{(2)}) \\
& =\lambda(Y_1^o((L(0) + xL(-1))v, x+z)w_{(1)} \otimes w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes Y_2((L(0) + (x-z)L(-1))v, x_1)w_{(2)}) \\
& =\lambda(Y_1^o((L(0) + xL(-1))v, x+z)w_{(1)} \otimes w_{(2)}) \\
& -\text{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)} \otimes Y_2((L(0) + xL(-1))v, x_1)w_{(2)}) \\
& = (Y'_{Q(z)}((L(0) + xL(-1))v, x)\lambda)(w_{(1)} \otimes w_{(2)}) \\
& = (Y'_{Q(z)}(L(0)v, x)\lambda)(w_{(1)} \otimes w_{(2)}) + (xY'_{Q(z)}(L(-1)v, x)\lambda)(w_{(1)} \otimes w_{(2)}),
\end{aligned}$$

proving (5.172). \square

Let W_3 also be an object of \mathcal{C} . Note that $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ acts on W'_3 in the natural way. The following result provides further motivation for the definition of our action (5.151) on $(W_1 \otimes W_2)^*$; recall the discussion preceding Proposition 5.22:

Proposition 5.55 *Let W_1 , W_2 and W_3 be generalized V -modules. Under the natural iso-*

morphism described in Remark 5.18 between the space of \tilde{A} -compatible linear maps

$$I : W_1 \otimes W_2 \rightarrow \overline{W_3}$$

and the space of \tilde{A} -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

determined by (5.125), the $Q(z)$ -intertwining maps I of type $\binom{W_3}{W_1 W_2}$ correspond exactly to the (grading restricted) \tilde{A} -compatible maps J that intertwine the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$$

and $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$.

Proof In view of (5.126), Remark 5.47 asserts that (5.146), or equivalently, (4.72), is equivalent to the condition

$$J \left(\tau_{W'_3} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) w'_{(3)} \right) = \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) J(w'_{(3)}), \quad (5.174)$$

that is, the condition that J intertwines the actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on W'_3 and on $(W_1 \otimes W_2)^*$ (recall (5.12)–(5.13)). Similarly, Remark 5.51 asserts that (4.73) is equivalent to the condition

$$J(L'(j)w'_{(3)}) = L'_{Q(z)}(j)J(w'_{(3)}) \quad (5.175)$$

for $j = -1, 0, 1$, that is, the condition that J intertwines the actions of $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$. \square

Notation 5.56 Given generalized V -modules W_1, W_2 and W_3 , we shall write $\mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$ for the space of (grading restricted) \tilde{A} -compatible linear maps

$$J : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

that intertwine the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$$

and $\mathfrak{sl}(2)$ on W'_3 and on $(W_1 \otimes W_2)^*$. Note that Proposition 5.55 gives a natural linear isomorphism

$$\begin{array}{ccc} \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3} & \xrightarrow{\sim} & \mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*} \\ I & \mapsto & J \end{array}$$

(recall from Definition 4.32 the notation for the space of $Q(z)$ -intertwining maps). As in Notation 5.23, we still use the symbol “prime” to denote this isomorphism in both directions:

$$\begin{aligned} \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3} &\xrightarrow{\sim} \mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*} \\ I &\mapsto I' \\ J' &\leftarrow J, \end{aligned}$$

so that in particular,

$$I'' = I \quad \text{and} \quad J'' = J$$

for $I \in \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}$ and $J \in \mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$, and the relation between I and I' is determined by

$$\langle w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle = I'(w'_{(3)})(w_{(1)} \otimes w_{(2)})$$

for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W'_3$, or equivalently,

$$w'_{(3)} \circ I = I'(w'_{(3)}).$$

Remark 5.57 Combining Proposition 5.55 with Proposition 4.38, we see that for any integer p , we also have a natural linear isomorphism

$$\mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*} \xrightarrow{\sim} \mathcal{V}_{W'_3 W_2}^{W'_1}$$

from $\mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$ to the space of logarithmic intertwining operators of type $(\begin{smallmatrix} W'_1 \\ W'_3 W_2 \end{smallmatrix})$. In particular, given any such logarithmic intertwining operator \mathcal{Y} and integer p , the map

$$(I_{\mathcal{Y}, p}^{Q(z)})' : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

defined by

$$(I_{\mathcal{Y}, p}^{Q(z)})'(w'_{(3)})(w_{(1)} \otimes w_{(2)}) = \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)} w_{(2)}) \rangle_{W'_1}$$

is \tilde{A} -compatible and intertwines both actions on both spaces.

We have formulated the notions of $Q(z)$ -product and $Q(z)$ -tensor product using $Q(z)$ -intertwining maps (Definitions 4.39 and 4.40). Now that we know that $Q(z)$ -intertwining maps can be interpreted as in Proposition 5.55 (and Notation 5.56), we can reformulate the notions of $Q(z)$ -product and $Q(z)$ -tensor product correspondingly (the proof of the next result is the same as that of Proposition 5.25):

Proposition 5.58 *Let \mathcal{C}_1 be either of the categories \mathcal{M}_{sg} or \mathcal{GM}_{sg} , as in Definition 4.39. For $W_1, W_2 \in \text{ob } \mathcal{C}_1$, a $Q(z)$ -product $(W_3; I_3)$ of W_1 and W_2 (recall Definition 4.39) amounts to an object (W_3, Y_3) of \mathcal{C}_1 equipped with a map $I'_3 \in \mathcal{N}[Q(z)]_{W'_3}^{(W_1 \otimes W_2)^*}$, that is, equipped with an \tilde{A} -compatible map*

$$I'_3 : W'_3 \rightarrow (W_1 \otimes W_2)^*$$

that intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ and of $\mathfrak{sl}(2)$. The map I'_3 corresponds to the $Q(z)$ -intertwining map

$$I_3 : W_1 \otimes W_2 \rightarrow \overline{W_3}$$

as above:

$$I'_3(w'_{(3)}) = w'_{(3)} \circ I_3$$

for $w'_{(3)} \in W'_3$ (recall 5.126)). Denoting this structure by $(W_3, Y_3; I'_3)$ or simply by $(W_3; I'_3)$, let $(W_4; I'_4)$ be another such structure. Then a morphism of $Q(z)$ -products from W_3 to W_4 amounts to a module map $\eta : W_3 \rightarrow W_4$ such that the diagram

$$\begin{array}{ccc} & (W_1 \otimes W_2)^* & \\ I'_4 \nearrow & & \nwarrow I'_3 \\ W'_4 & \xrightarrow{\eta'} & W'_3 \end{array}$$

commutes, where η' is the natural map given by (2.99). \square

Corollary 5.59 *Let \mathcal{C} be a full subcategory of either \mathcal{M}_{sg} or \mathcal{GM}_{sg} , as in Definition 4.40. For $W_1, W_2 \in \text{ob } \mathcal{C}$, a $Q(z)$ -tensor product $(W_0; I_0)$ of W_1 and W_2 in \mathcal{C} , if it exists, amounts to an object $W_0 = W_1 \boxtimes_{Q(z)} W_2$ of \mathcal{C} and a structure $(W_0 = W_1 \boxtimes_{Q(z)} W_2; I'_0)$ as in Proposition 5.58, with*

$$I'_0 : (W_1 \boxtimes_{Q(z)} W_2)' \longrightarrow (W_1 \otimes W_2)^*$$

in $\mathcal{N}[Q(z)]^{(W_1 \otimes W_2)^*}_{(W_1 \boxtimes_{Q(z)} W_2)'}$, such that for any such pair $(W; I')$ ($W \in \text{ob } \mathcal{C}$), with

$$I' : W' \longrightarrow (W_1 \otimes W_2)^*$$

in $\mathcal{N}[Q(z)]^{(W_1 \otimes W_2)^*}_{W'}$, there is a unique module map

$$\chi : W' \longrightarrow (W_1 \boxtimes_{Q(z)} W_2)'$$

such that the diagram

$$\begin{array}{ccc} & (W_1 \otimes W_2)^* & \\ I' \nearrow & & \nwarrow I'_0 \\ W' & \xrightarrow{\chi} & (W_1 \boxtimes_{Q(z)} W_2)' \end{array}$$

commutes. Here $\chi = \eta'$, where η is a correspondingly unique module map

$$\eta : W_1 \boxtimes_{Q(z)} W_2 \longrightarrow W.$$

Also, the map I'_0 , which is \tilde{A} -compatible and which intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ and of $\mathfrak{sl}(2)$, is related to the $Q(z)$ -intertwining map

$$I_0 = \boxtimes_{Q(z)} : W_1 \otimes W_2 \longrightarrow \overline{W_1 \boxtimes_{Q(z)} W_2}$$

by

$$I'_0(w') = w' \circ \boxtimes_{Q(z)}$$

for $w' \in (W_1 \boxtimes_{Q(z)} W_2)'$, that is,

$$I'_0(w')(w_{(1)} \otimes w_{(2)}) = \langle w', w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, using the notation (4.80). \square

Definition 5.60 For $W_1, W_2 \in \text{ob } \mathcal{C}$, define the subset

$$W_1 \boxtimes_{Q(z)} W_2 \subset (W_1 \otimes W_2)^*$$

of $(W_1 \otimes W_2)^*$ to be the union of the images

$$I'(W') \subset (W_1 \otimes W_2)^*$$

as $(W; I)$ ranges through all the $Q(z)$ -products of W_1 and W_2 with $W \in \text{ob } \mathcal{C}$. Equivalently, $W_1 \boxtimes_{Q(z)} W_2$ is the union of the images $I'(W')$ as W (or W') ranges through $\text{ob } \mathcal{C}$ and I' ranges through $\mathcal{N}[Q(z)]_{W'}^{(W_1 \otimes W_2)^*}$ —the space of \tilde{A} -compatible linear maps

$$W' \rightarrow (W_1 \otimes W_2)^*$$

intertwining the actions of both

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$$

and $\mathfrak{sl}(2)$ on both spaces.

Remark 5.61 Since \mathcal{C} is closed under direct sums (Assumption 5.28), it is clear that $W_1 \boxtimes_{Q(z)} W_2$ is in fact a linear subspace of $(W_1 \otimes W_2)^*$, and in particular, it can be defined alternatively as the sum of all the images $I'(W')$:

$$W_1 \boxtimes_{Q(z)} W_2 = \sum I'(W') = \bigcup I'(W') \subset (W_1 \otimes W_2)^*, \quad (5.176)$$

where the sum and union both range over $W \in \text{ob } \mathcal{C}$, $I \in \mathcal{M}[Q(z)]_{W_1 W_2}^W$.

For any generalized V -modules W_1 and W_2 , using the operator $L'_{Q(z)}(0)$ (recall (5.169)) on $(W_1 \otimes W_2)^*$ we define the generalized $L'_{Q(z)}(0)$ -eigenspaces $((W_1 \otimes W_2)^*)_{[n]; Q(z)}$ for $n \in \mathbb{C}$ in the usual way:

$$((W_1 \otimes W_2)^*)_{[n]; Q(z)} = \{w \in (W_1 \otimes W_2)^* \mid (L'_{Q(z)}(0) - n)^m w = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large}\}. \quad (5.177)$$

Then we have the (proper) subspace

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{[n]; Q(z)} \subset (W_1 \otimes W_2)^*. \quad (5.178)$$

We also define the ordinary $L'_{Q(z)}(0)$ -eigenspaces $((W_1 \otimes W_2)^*)_{(n);Q(z)}$ in the usual way:

$$((W_1 \otimes W_2)^*)_{(n);Q(z)} = \{w \in (W_1 \otimes W_2)^* \mid L'_{P(z)}(0)w = nw\}. \quad (5.179)$$

Then we have the (proper) subspace

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{(n);Q(z)} \subset (W_1 \otimes W_2)^*. \quad (5.180)$$

Just as in Proposition 5.31, we have:

Proposition 5.62 *Let $W_1, W_2 \in \text{ob } \mathcal{C}$.*

(a) *The elements of $W_1 \boxtimes_{Q(z)} W_2$ are exactly the linear functionals on $W_1 \otimes W_2$ of the form $w' \circ I(\cdot \otimes \cdot)$ for some $Q(z)$ -intertwining map I of type $\binom{W}{W_1 W_2}$ and some $w' \in W'$, $W \in \text{ob } \mathcal{C}$.*

(b) *Let $(W; I)$ be any $Q(z)$ -product of W_1 and W_2 , with W any generalized V -module. Then for $n \in \mathbb{C}$,*

$$I'(W'_{[n]}) \subset ((W_1 \otimes W_2)^*)_{[n];Q(z)}$$

and

$$I'(W'_{(n)}) \subset ((W_1 \otimes W_2)^*)_{(n);Q(z)}.$$

(c) *The structure $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ (recall (5.150)) satisfies all the axioms in the definition of (strongly \tilde{A} -graded) generalized V -module except perhaps for the two grading conditions (2.85) and (2.86).*

(d) *Suppose that the objects of the category \mathcal{C} consist only of (strongly \tilde{A} -graded) ordinary, as opposed to generalized, V -modules. Then the structure $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ satisfies all the axioms in the definition of (strongly \tilde{A} -graded ordinary) V -module except perhaps for (2.85) and (2.86).*

Proof Part (a) is clear from the definition of $W_1 \boxtimes_{Q(z)} W_2$, and (b) follows from (5.175) with $j = 0$.

To prove (c), let $(W; I)$ be any $Q(z)$ -product of W_1 and W_2 , with W any generalized V -module. Then $(I'(W'), Y'_{Q(z)})$ satisfies all the conditions in the definition of (strongly \tilde{A} -graded) generalized V -module since I' is \tilde{A} -compatible and intertwines the actions of $V \otimes \mathbb{C}[t, t^{-1}]$ and of $\mathfrak{sl}(2)$; the \mathbb{C} -grading follows from Part (b). Since $W_1 \boxtimes_{Q(z)} W_2$ is the sum of these structures $I'(W')$ over $W \in \text{ob } \mathcal{C}$ (recall (5.129)), $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ satisfies all the conditions in the definition of generalized module except perhaps for (2.85) and (2.86).

Part (d) is proved by the same argument as for (c): For $(W; I)$ any $Q(z)$ -product of possibly generalized V -modules W_1 and W_2 , with W any ordinary V -module, $(I'(W'), Y'_{Q(z)})$ satisfies all the conditions in the definition of (strongly \tilde{A} -graded) ordinary V -module; the \mathbb{C} -grading (by ordinary $L'_{Q(z)}(0)$ -eigenspaces) again follows from Part (b). \square

We now have the following generalization of Proposition 5.8 in [HL5], characterizing $W_1 \boxtimes_{Q(z)} W_2$, including its existence, in terms of $W_1 \boxtimes_{Q(z)} W_2$; the proof is the same as that of Proposition 5.32:

Proposition 5.63 *Let $W_1, W_2 \in \text{ob } \mathcal{C}$. If $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ is an object of \mathcal{C} , denote by $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)})$ its contragredient module. Then the $Q(z)$ -tensor product of W_1 and W_2 in \mathcal{C} exists and is $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; i')$, where i is the natural inclusion from $W_1 \boxtimes_{Q(z)} W_2$ to $(W_1 \otimes W_2)^*$ (recall Notation 5.56). Conversely, let us assume that \mathcal{C} is closed under quotients. If the $Q(z)$ -tensor product of W_1 and W_2 in \mathcal{C} exists, then $(W_1 \boxtimes_{Q(z)} W_2, Y'_{Q(z)})$ is an object of \mathcal{C} . \square*

Remark 5.64 Suppose that $W_1 \boxtimes_{Q(z)} W_2$ is an object of \mathcal{C} . From Corollary 5.59 and Proposition 5.63 we see that

$$\langle \lambda, w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle \Big|_{W_1 \boxtimes_{Q(z)} W_2} = \lambda(w_{(1)} \otimes w_{(2)}) \quad (5.181)$$

for $\lambda \in W_1 \boxtimes_{Q(z)} W_2 \subset (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

As in the $P(z)$ -case, our next goal is to present an alternative description of the subspace $W_1 \boxtimes_{Q(z)} W_2$ of $(W_1 \otimes W_2)^*$. The main ingredient of this description will be the “ $Q(z)$ -compatibility condition,” as was the case in [HL5]–[HL6].

Take W_1 and W_2 to be arbitrary generalized V -modules. Let (W, I) (W a generalized V -module) be a $Q(z)$ -product of W_1 and W_2 and let $w' \in W'$. Then from (5.174), Proposition 5.58, (5.124), (5.7) and (5.150), we have, for all $v \in V$,

$$\begin{aligned} & \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) I'(w') \\ &= I' \left(\tau_{W'} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) w' \right) \\ &= I' \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_{W'}(v, x_0) w' \right) \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) I'(Y_{W'}(v, x_0) w') \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) I'(\tau_{W'}(Y_t(v, x_0)) w') \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \tau_{Q(z)}(Y_t(v, x_0)) I'(w') \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_{Q(z)}(v, x_0) I'(w'). \end{aligned} \quad (5.182)$$

That is, $I'(w')$ satisfies the following nontrivial and subtle condition on $\lambda \in (W_1 \otimes W_2)^*$:

The $Q(z)$ -compatibility condition

- (a) The $Q(z)$ -lower truncation condition: For all $v \in V$, the formal Laurent series $Y'_{Q(z)}(v, x)\lambda$ involves only finitely many negative powers of x .

(b) The following formula holds:

$$\begin{aligned} \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda \\ = z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_{Q(z)}(v, x_0) \lambda \quad \text{for all } v \in V. \end{aligned} \quad (5.183)$$

(Note that the two sides of (5.183) are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2)^*$. Note also that Condition (a) insures that the right-hand side in Condition (b) is well defined.)

Notation 5.65 Note that the set of elements of $(W_1 \otimes W_2)^*$ satisfying either the full $Q(z)$ -compatibility condition or Part (a) of this condition forms a subspace. We shall denote the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $Q(z)$ -compatibility condition by

$$\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*).$$

We know that each space $((W_1 \otimes W_2)^*)^{(\beta)}$ is $L'_{Q(z)}(0)$ -stable (recall Proposition 5.48 and Remark 5.52), so that we may consider the subspaces

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{[n]; Q(z)}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

and

$$\coprod_{n \in \mathbb{C}} ((W_1 \otimes W_2)^*)_{(n); Q(z)}^{(\beta)} \subset ((W_1 \otimes W_2)^*)^{(\beta)}$$

(recall Remark 2.13). We define the two subspaces

$$((W_1 \otimes W_2)^*)_{[\mathbb{C}]; Q(z)}^{(\tilde{A})} = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{[n]; Q(z)}^{(\beta)} \subset (W_1 \otimes W_2)^* \quad (5.184)$$

and

$$((W_1 \otimes W_2)^*)_{(\mathbb{C}); Q(z)}^{(\tilde{A})} = \coprod_{n \in \mathbb{C}} \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{(n); Q(z)}^{(\beta)} \subset (W_1 \otimes W_2)^*. \quad (5.185)$$

Remark 5.66 Any $L'_{Q(z)}(0)$ -stable subspace of $((W_1 \otimes W_2)^*)_{[\mathbb{C}]; Q(z)}^{(\tilde{A})}$ is graded by generalized eigenspaces (again recall Remark 2.13), and if such a subspace is also \tilde{A} -graded, then it is doubly graded; similarly for subspaces of $((W_1 \otimes W_2)^*)_{(\mathbb{C}); Q(z)}^{(\tilde{A})}$.

We have:

Lemma 5.67 *Suppose that $\lambda \in ((W_1 \otimes W_2)^*)_{[\mathbb{C}]; Q(z)}^{(\tilde{A})}$ satisfies the $Q(z)$ -compatibility condition. Then every \tilde{A} -homogeneous component of λ also satisfies this condition.*

Proof When $v \in V$ is \tilde{A} -homogeneous,

$$\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(v, x_0)\right) \quad \text{and} \quad z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y'_{Q(z)}(v, x_0)$$

are both \tilde{A} -homogeneous as operators. By comparing the \tilde{A} -homogeneous components of both sides of (5.183), we see that the \tilde{A} -homogeneous components of λ also satisfy the $Q(z)$ -compatibility condition. \square

Remark 5.68 Just as in Remark 5.37, note that both the spaces $((W_1 \otimes W_2)^*)_{[\mathbb{C}];Q(z)}^{(\tilde{A})}$ and $((W_1 \otimes W_2)^*)_{(\mathbb{C});Q(z)}^{(\tilde{A})}$ are stable under the component operators $\tau_{Q(z)}(v \otimes t^m)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$; this uses Proposition 5.48, Remark 5.52, Propositions 5.49 and 5.50, and (5.172).

Again let $(W; I)$ (W a generalized V -module) be a $Q(z)$ -product of W_1 and W_2 and let $w' \in W'$. Since I' in particular intertwines the actions of $V \otimes \mathbb{C}[t, t^{-1}]$ and of $\mathfrak{sl}(2)$, and is \tilde{A} -compatible, $I'(W')$ is a generalized V -module (recall the proof of Proposition 5.62). Thus for every $w' \in W'$, $I'(w')$ also satisfies the following condition on $\lambda \in (W_1 \otimes W_2)^*$:

The $Q(z)$ -local grading restriction condition

(a) The $Q(z)$ -grading condition: λ is a (finite) sum of generalized eigenvectors for the operator $L'_{Q(z)}(0)$ on $(W_1 \otimes W_2)^*$ that are also homogeneous with respect to \tilde{A} , that is,

$$\lambda \in ((W_1 \otimes W_2)^*)_{[\mathbb{C}];Q(z)}^{(\tilde{A})}.$$

(b) Let $W_{\lambda;Q(z)}$ be the smallest doubly graded (or equivalently, \tilde{A} -graded; recall Remark 5.66) subspace of $((W_1 \otimes W_2)^*)_{[\mathbb{C}];Q(z)}^{(\tilde{A})}$ containing λ and stable under the component operators $\tau_{Q(z)}(v \otimes t^m)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$. (In view of Remark 5.68, $W_{\lambda;Q(z)}$ indeed exists.) Then $W_{\lambda;Q(z)}$ has the properties

$$\dim(W_{\lambda})_{[n];Q(z)}^{(\beta)} < \infty, \tag{5.186}$$

$$(W_{\lambda})_{[n+k];Q(z)}^{(\beta)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative,} \tag{5.187}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where as usual the subscripts denote the \mathbb{C} -grading and the superscripts denote the \tilde{A} -grading.

In the case that W is an (ordinary) V -module and $w' \in W'$, $I'(w')$ also satisfies the following $L(0)$ -semisimple version of this condition on $\lambda \in (W_1 \otimes W_2)^*$:

The $L(0)$ -semisimple $Q(z)$ -local grading restriction condition

(a) The $L(0)$ -semisimple $Q(z)$ -grading condition: λ is a (finite) sum of eigenvectors for the operator $L'_{Q(z)}(0)$ on $(W_1 \otimes W_2)^*$ that are also homogeneous with respect to \tilde{A} , that is,

$$\lambda \in ((W_1 \otimes W_2)^*)_{(\mathbb{C});Q(z)}^{(\tilde{A})}.$$

(b) Consider $W_{\lambda;Q(z)}$ as above, which in this case is in fact the smallest doubly graded (or equivalently, \tilde{A} -graded) subspace of $((W_1 \otimes W_2)^*)_{(\mathbb{C});Q(z)}^{(\tilde{A})}$ containing λ and stable under the component operators $\tau_{Q(z)}(v \otimes t^m)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$. Then $W_{\lambda;Q(z)}$ has the properties

$$\dim(W_{\lambda;Q(z)})_{(n);Q(z)}^{(\beta)} < \infty, \quad (5.188)$$

$$(W_{\lambda})_{(n+k);Q(z)}^{(\beta)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative,} \quad (5.189)$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the \mathbb{C} -grading and the superscripts denote the \tilde{A} -grading.

Notation 5.69 Note that the set of elements of $(W_1 \otimes W_2)^*$ satisfying either of these two $Q(z)$ -local grading restriction conditions, or either of the Part (a)'s in these conditions, forms a subspace. We shall denote the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $Q(z)$ -local grading restriction condition and the $L(0)$ -semisimple $Q(z)$ -local grading restriction condition by

$$\text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*)$$

and

$$\text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*),$$

respectively.

We have the following important theorems generalizing the corresponding results stated in [HL5] and proved in [HL6]. The proofs of these theorems will be given in the next section.

Theorem 5.70 *Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $Q(z)$ -compatibility condition. Then when acting on λ , the Jacobi identity for $Y'_{Q(z)}$ holds, that is,*

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{Q(z)}(u, x_1) Y'_{Q(z)}(v, x_2) \lambda \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{Q(z)}(v, x_2) Y'_{Q(z)}(u, x_1) \lambda \\ & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{Q(z)}(Y(u, x_0)v, x_2) \lambda \end{aligned} \quad (5.190)$$

for $u, v \in V$.

Theorem 5.71 *The subspace $\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*)$ of $(W_1 \otimes W_2)^*$ is stable under the operators $\tau_{Q(z)}(v \otimes t^n)$ for $v \in V$ and $n \in \mathbb{Z}$, and in the Möbius case, also under the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$; similarly for the subspaces $\text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*$ and $\text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*$.*

We have:

Theorem 5.72 *The space $\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*)$, equipped with the vertex operator map $Y'_{Q(z)}$ and, in case V is Möbius, also equipped with the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$, is a weak V -module; similarly for the spaces*

$$(\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*)) \cap (\text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*))$$

and

$$(\text{COMP}_{Q(z)}((W_1 \otimes W_2)^*)) \cap (\text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*)).$$

Proof By Theorem 5.71, $Y'_{Q(z)}$ is a map from the tensor product of V with any of these three subspaces to the space of formal Laurent series with elements of the subspace as coefficients. By Proposition 5.49 and Theorem 5.70 and, in the case that V is Möbius, also by Propositions 5.53 and 5.54, we see that all the axioms for weak V -module are satisfied. \square

Moreover, we have the following consequence of Theorem 5.72 and Lemma 5.67, just as in Theorem 5.44:

Theorem 5.73 *Let*

$$\lambda \in \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}];Q(z)}((W_1 \otimes W_2)^*).$$

Then $W_{\lambda;Q(z)}$ (recall Part (b) of the $Q(z)$ -local grading restriction condition) equipped with the vertex operator map $Y'_{Q(z)}$ and, in case V is Möbius, also equipped with the operators $L'_{Q(z)}(-1)$, $L'_{Q(z)}(0)$ and $L'_{Q(z)}(1)$, is a (strongly-graded) generalized V -module. If in addition

$$\lambda \in \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C});Q(z)}((W_1 \otimes W_2)^*),$$

that is, λ is a sum of eigenvectors of $L'_{Q(z)}(0)$, then $W_{\lambda;Q(z)} (\subset ((W_1 \otimes W_2)^)_{(\mathbb{C});Q(z)}^{(\bar{A})})$ is a (strongly-graded) V -module. \square*

Finally, as in Theorem 5.45, we can give an alternative description of $W_1 \boxtimes_{Q(z)} W_2$ by characterizing the elements of $W_1 \boxtimes_{Q(z)} W_2$ using the $Q(z)$ -compatibility condition and the $Q(z)$ -local grading restriction conditions, generalizing Theorem 6.3 in [HL5]. The proof of the following theorem is the same as that of Theorem 5.45.

Theorem 5.74 *Suppose that for every element*

$$\lambda \in \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; Q(z)}((W_1 \otimes W_2)^*)$$

the (strongly-graded) generalized module $W_{\lambda; Q(z)}$ given in Theorem 5.73 is an object of \mathcal{C} (this of course holds in particular if \mathcal{C} is \mathcal{GM}_{sg}). Then

$$W_1 \boxtimes_{Q(z)} W_2 = \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{[\mathbb{C}]; Q(z)}((W_1 \otimes W_2)^*).$$

Suppose that \mathcal{C} is a category of strongly-graded V -modules (that is, $\mathcal{C} \subset \mathcal{M}_{sg}$) and that for every element

$$\lambda \in \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}); Q(z)}((W_1 \otimes W_2)^*)$$

the (strongly-graded) V -module $W_{\lambda; Q(z)}$ given in Theorem 5.73 is an object of \mathcal{C} (which of course holds in particular if \mathcal{C} is \mathcal{M}_{sg}). Then

$$W_1 \boxtimes_{Q(z)} W_2 = \text{COMP}_{Q(z)}((W_1 \otimes W_2)^*) \cap \text{LGR}_{(\mathbb{C}); Q(z)}((W_1 \otimes W_2)^*). \quad \square$$

6 Proof of the theorems used in the constructions

The primary goal of this section is to prove Theorems 5.39, 5.40, 5.70 and 5.71. In Subsection 6.1 we prove Theorems 5.39 and 5.40, and in Subsection 6.2, Theorems 5.70 and 5.71. The proofs in Subsection 6.1 are new, even for the category of (ordinary) modules for a vertex operator algebra satisfying the finiteness and reductivity conditions treated in [HL5]–[HL7]. In [HL5]–[HL7], for a vertex operator algebra satisfying these conditions, Theorems 5.70 and 5.71, in the $Q(z)$ case, were proved first, and then Theorems 5.39 and 5.40, in the $P(z)$ case, were proved using results from the $Q(z^{-1})$ case and relations between $P(z)$ -tensor products and $Q(z^{-1})$ -tensor products. In Subsection 6.1, we prove Theorems 5.39 and 5.40 directly, without using any results from the $Q(z)$ case. As usual, the reader should observe the justifiability of each step in the arguments (the well-definedness of the formal series, etc.); again as usual, this is sometimes quite subtle.

6.1 Proofs of Theorems 5.39 and 5.40

We first prove a formula for vertex operators that will be needed in the proofs of both Theorem 5.39 and Theorem 5.40.

Lemma 6.1 *For $u, v \in V$, we have*

$$\begin{aligned} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, -x_0 x_1^{-1} x_2^{-1}) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} v \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} Y(u, x_0) v. \end{aligned} \quad (6.1)$$

Proof Using (3.59), (3.60), (3.65) and (2.11), we have

$$\begin{aligned} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, -x_0 x_1^{-1} x_2^{-1}) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) e^{x_2 L(1)} Y(e^{-x_2(1-x_0 x_1^{-1}) L(1)} (1 - x_0 x_1^{-1})^{-2L(0)} \cdot \\ \cdot e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, -x_0 x_1^{-1} x_2^{-1} (1 - x_0 x_1^{-1})^{-1}) (-x_2^{-2})^{L(0)} \\ = x_2^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{x_1^{-1}} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} \cdot \\ \cdot Y((-x_2^2)^{L(0)} e^{-x_2(1-x_0 x_1^{-1}) L(1)} (1 - x_0 x_1^{-1})^{-2L(0)} \cdot \\ \cdot e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, x_0 x_1^{-1} (x_2^{-1} - x_0 x_1^{-1} x_2^{-1})^{-1}) \\ = x_2^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{x_1^{-1}} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)} \cdot \\ \cdot Y(e^{x_2^{-1}(1-x_0 x_1^{-1}) L(1)} (-x_2^2)^{L(0)} (1 - x_0 x_1^{-1})^{-2L(0)} e^{x_1 L(1)} (-x_1^{-2})^{L(0)} u, x_0) \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) e^{x_2 L(1)} (-x_2^{-2})^{L(0)}. \end{aligned}$$

$$\begin{aligned}
& \cdot Y(e^{x_2^{-1}(1-x_0x_1^{-1})L(1)}(-x_2^{-1}(1-x_0x_1^{-1}))^{-2})^{L(0)}e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0) \\
&= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)e^{x_2L(1)}(-x_2^{-2})^{L(0)} \cdot \\
& \cdot Y(e^{x_2^{-1}(1-x_0x_1^{-1})L(1)}e^{-x_1x_2^{-2}(1-x_0x_1^{-1})^2L(1)}(-x_2^{-1}(1-x_0x_1^{-1}))^{-2})^{L(0)}(-x_1^{-2})^{L(0)}u, x_0) \\
&= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)e^{x_2L(1)}(-x_2^{-2})^{L(0)} \cdot \\
& \cdot Y(e^{x_2^{-1}(1-x_0x_1^{-1})(1-x_2^{-1}(x_1-x_0))L(1)}(x_2^{-1}(x_1-x_0))^{-2})^{L(0)}u, x_0) \\
&= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)e^{x_2L(1)}(-x_2^{-2})^{L(0)}Y(u, x_0).
\end{aligned}$$

□

Proof of Theorem 5.39 Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition, that is, satisfying (a) the $P(z)$ -lower truncation condition—for all $v \in V$, the formal Laurent series $Y'_{P(z)}(v, x)\lambda$ involves only finitely many negative powers of x , and (b) formula (5.136) for all $v \in V$.

For $u, v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, by definition,

$$\begin{aligned}
& \left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y'_{P(z)}(u, x_1)Y'_{P(z)}(v, x_2)\lambda\right)(w_{(1)} \otimes w_{(2)}) \\
&= x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\left((Y'_{P(z)}(v, x_2)\lambda)(w_{(1)} \otimes Y_2^o(u, x_1)w_{(2)})\right. \\
& \quad \left.+ \text{Res}_{y_1} z^{-1}\delta\left(\frac{x_1^{-1}-y_1}{z}\right)(Y'_{P(z)}(v, x_2)\lambda)(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)})\right) \\
&= x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\left(\lambda(w_{(1)} \otimes Y_2^o(v, x_2)Y_2^o(u, x_1)w_{(2)})\right. \\
& \quad \left.+ \text{Res}_{y_2} z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right)\lambda(Y_1(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v, y_2)w_{(1)} \otimes Y_2^o(u, x_1)w_{(2)})\right. \\
& \quad \left.+ \text{Res}_{y_1} z^{-1}\delta\left(\frac{x_1^{-1}-y_1}{z}\right)(Y'_{P(z)}(v, x_2)\lambda)(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)})\right) \\
&= x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\lambda(w_{(1)} \otimes Y_2^o(v, x_2)Y_2^o(u, x_1)w_{(2)}) \\
& \quad + x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\text{Res}_{y_2} z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v, y_2)w_{(1)} \otimes Y_2^o(u, x_1)w_{(2)}) \\
& \quad + x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\text{Res}_{y_1} z^{-1}\delta\left(\frac{x_1^{-1}-y_1}{z}\right) \cdot \\
& \quad \cdot (Y'_{P(z)}(v, x_2)\lambda)(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, y_1)w_{(1)} \otimes w_{(2)}). \tag{6.2}
\end{aligned}$$

Using (2.6) and (5.136), we see that the third term on the right-hand side of (6.2) is equal to

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_2^{-1} - x_1^{-1}}{x_0 x_1^{-1} x_2^{-1}} \right) \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot \\
& \quad \cdot (Y'_{P(z)}(v, x_2) \lambda) (Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{y_1} x_1^{-1} x_2^{-1} (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{x_2^{-1} - y_1 - z}{x_0 x_1^{-1} x_2^{-1}} \right) z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot \\
& \quad \cdot (Y'_{P(z)}(v, x_2) \lambda) (Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{y_1} x_1^{-1} x_2^{-1} (x_0 x_1^{-1} x_2^{-1} + y_1)^{-1} \delta \left(\frac{x_2^{-1} - z}{x_0 x_1^{-1} x_2^{-1} + y_1} \right) z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot \\
& \quad \cdot (Y'_{P(z)}(v, x_2) \lambda) (Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{y_1} x_1^{-1} x_2^{-1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot \\
& \quad \cdot \left(\tau_{P(z)} \left((x_0 x_1^{-1} x_2^{-1} + y_1)^{-1} \delta \left(\frac{x_2^{-1} - z}{x_0 x_1^{-1} x_2^{-1} + y_1} \right) Y_t(v, x_2) \right) \lambda \right) \\
& \quad (Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = \text{Res}_{y_1} x_1^{-1} x_2^{-1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot \\
& \quad \cdot \left(z^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1} - y_1}{z} \right) \cdot \right. \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, x_0 x_1^{-1} x_2^{-1} + y_1) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& \quad + (x_0 x_1^{-1} x_2^{-1} + y_1)^{-1} \delta \left(\frac{z - x_2^{-1}}{-x_0 x_1^{-1} x_2^{-1} - y_1} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}) \Big) \\
& = \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{z + y_1}{x_1^{-1}} \right) (z + y_1)^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{z + y_1} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, x_0 x_1^{-1} x_2^{-1} + y_1) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& \quad + \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{z + y_1}{x_1^{-1}} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{z + y_1 - x_2^{-1}}{-x_0 x_1^{-1} x_2^{-1}} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}). \tag{6.3}
\end{aligned}$$

By (2.11) and (2.6), the right-hand side of (6.3) is equal to

$$\begin{aligned}
& \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{z + y_1}{x_1^{-1}} \right) x_1 \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{x_1^{-1}} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, x_0 x_1^{-1} x_2^{-1} + y_1) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)})
\end{aligned}$$

$$\begin{aligned}
& + \text{Res}_{y_1} x_2^{-1} \delta \left(\frac{z + y_1}{x_1^{-1}} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{x_1^{-1} - x_2^{-1}}{-x_0 x_1^{-1} x_2^{-1}} \right) \\
& \quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}) \\
& = \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, x_0 x_1^{-1} x_2^{-1} + y_1) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& + \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \\
& \quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}). \tag{6.4}
\end{aligned}$$

Since

$$\text{Res}_{y_2} y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) = 1,$$

the first term on the right-hand side of (6.4) can be written as

$$\begin{aligned}
& \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_2} y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, x_0 x_1^{-1} x_2^{-1} + y_1) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{y_2 - y_1}{x_0 x_1^{-1} x_2^{-1}} \right) \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}), \tag{6.5}
\end{aligned}$$

where we have also used (2.11) and (2.6). Again using (2.6) and (2.11), we see that the right-hand side of (6.5) is also equal to

$$\begin{aligned}
& x_2^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{x_1^{-1}} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = x_2^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1}}{x_1^{-1}} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - x_0 x_1^{-1} x_2^{-1} - y_1}{z} \right) \\
& \quad \cdot y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) y_2^{-1} \delta \left(\frac{x_0 x_1^{-1} x_2^{-1} + y_1}{y_2} \right) \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{y_2 - y_1}{x_0 x_1^{-1} x_2^{-1}} \right) \\
& \quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}). \tag{6.6}
\end{aligned}$$

That is, in the middle delta-function expression in the right-hand side of (6.5), we may replace x_1 by x_2 and y_1 by y_2 .

From (6.2)–(6.6), we obtain

$$\begin{aligned}
& \left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
&= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \lambda(w_{(1)} \otimes Y_2^o(v, x_2) Y_2^o(u, x_1) w_{(2)}) \\
&\quad + x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) w_{(1)} \otimes Y_2^o(u, x_1) w_{(2)}) \\
&\quad + x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{y_2 - y_1}{x_0 x_1^{-1} x_2^{-1}} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes w_{(2)}) \\
&\quad + x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}) \Big). \tag{6.7}
\end{aligned}$$

From (6.6) and (6.7), replacing u, v, x_1, x_2, x_0 by $v, u, x_2, x_1, -x_0$, respectively, and also using (2.6), we find that

$$\begin{aligned}
& \left(-x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{P(z)}(v, x_2) Y'_{P(z)}(u, x_1) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
&= -x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2^o(u, x_1) Y_2^o(v, x_2) w_{(2)}) \\
&\quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_1} z^{-1} \delta \left(\frac{x_1^{-1} - y_1}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) w_{(1)} \otimes Y_2^o(v, x_2) w_{(2)}) \\
&\quad - x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) (x_0 x_1^{-1} x_2^{-1})^{-1} \delta \left(\frac{y_1 - y_2}{-x_0 x_1^{-1} x_2^{-1}} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} u, y_1) Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) w_{(1)} \otimes w_{(2)}) \\
&\quad - x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)} v, y_2) w_{(1)} \otimes Y_2^o(u, x_1) w_{(2)}) \Big). \tag{6.8}
\end{aligned}$$

Using (6.7), (6.8), the Jacobi identity, the opposite Jacobi identity (2.61) and (2.6), we obtain

$$\left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2) \lambda \right)$$

$$\begin{aligned}
& -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y'_{P(z)}(v,x_2)Y'_{P(z)}(u,x_1)\lambda\Big(w_{(1)}\otimes w_{(2)}\Big) \\
& = \lambda\Big(w_{(1)}\otimes\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_2^o(v,x_2)Y_2^o(u,x_1)\right. \\
& \quad \left.-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_2^o(u,x_1)Y_2^o(v,x_2)\right)w_{(2)}\Big) \\
& \quad +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right) \\
& \quad \cdot\lambda\Bigg(\left((x_0x_1^{-1}x_2^{-1})^{-1}\delta\left(\frac{y_2-y_1}{x_0x_1^{-1}x_2^{-1}}\right)\right. \\
& \quad \cdot Y_1(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,y_1) \\
& \quad \left.-(x_0x_1^{-1}x_2^{-1})^{-1}\delta\left(\frac{y_1-y_2}{-x_0x_1^{-1}x_2^{-1}}\right)\right. \\
& \quad \left.\cdot Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,y_1)Y_1(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)\right)w_{(1)}\otimes w_{(2)}\Bigg) \\
& = \lambda\Big(w_{(1)}\otimes\left(x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_2^o(Y(u,x_0)v,x_2)\right)w_{(2)}\Big) \\
& \quad +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right) \\
& \quad \cdot\lambda\Bigg(\left(y_2^{-1}\delta\left(\frac{y_1+x_0x_1^{-1}x_2^{-1}}{y_2}\right)\right. \\
& \quad \cdot Y_1(Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,-x_0x_1^{-1}x_2^{-1})e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)\Bigg)w_{(1)}\otimes w_{(2)}\Bigg) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(w_{(1)}\otimes Y_2^o(Y(u,x_0)v,x_2)w_{(2)}) \\
& \quad +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right) \\
& \quad \cdot\lambda\Bigg(\left(y_1^{-1}\delta\left(\frac{y_2-x_0x_1^{-1}x_2^{-1}}{y_1}\right)\right. \\
& \quad \cdot Y_1(Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,-x_0x_1^{-1}x_2^{-1})e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)\Bigg)w_{(1)}\otimes w_{(2)}\Bigg) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(w_{(1)}\otimes Y_2^o(Y(u,x_0)v,x_2)w_{(2)}) \\
& \quad +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_2}z^{-1}\delta\left(\frac{x_2^{-1}-y_2}{z}\right) \\
& \quad \cdot\lambda(Y_1(Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u,-x_0x_1^{-1}x_2^{-1})e^{x_2L(1)}(-x_2^{-2})^{L(0)}v,y_2)w_{(1)}\otimes w_{(2)})
\end{aligned} \tag{6.9}$$

Finally, from (6.1) we see that the right-hand side of (6.9) becomes

$$\begin{aligned}
& x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(u, x_0)v, x_2)w_{(2)}) \\
& + x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_2} z^{-1} \delta \left(\frac{x_2^{-1} - y_2}{z} \right) \cdot \\
& \cdot \lambda(Y_1(e^{x_2 L(1)}(-x_2^{-2})^{L(0)}Y(u, x_0)v, y_2)w_{(1)} \otimes w_{(2)}) \\
& = \left(x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{P(z)}(Y(u, x_0)v, x_2)\lambda \right) (w_{(1)} \otimes w_{(2)}), \tag{6.10}
\end{aligned}$$

and we have proved the Jacobi identity and hence Theorem 5.39. \square

Proof of Theorem 5.40 Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition. We first want to prove that the coefficient of each power of x in $Y'_{P(z)}(u, x_0)Y'_{P(z)}(v, x)\lambda$ is a formal Laurent series involving only finitely many negative powers of x_0 and that

$$\begin{aligned}
& \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(u, x_1) \right) Y'_{P(z)}(v, x)\lambda \\
& = x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(u, x_1)Y'_{P(z)}(v, x)\lambda \tag{6.11}
\end{aligned}$$

for all $u, v \in V$. Using the commutator formula (Proposition 5.9) for $Y'_{P(z)}$, we have

$$\begin{aligned}
& Y'_{P(z)}(u, x_0)Y'_{P(z)}(v, x)\lambda \\
& = Y'_{P(z)}(v, x)Y'_{P(z)}(u, x_0)\lambda \\
& \quad - \text{Res}_y x_0^{-1} \delta \left(\frac{x - y}{x_0} \right) Y'_{P(z)}(Y(v, y)u, x_0)\lambda. \tag{6.12}
\end{aligned}$$

Each coefficient in x of the right-hand side of (6.12) is a formal Laurent series involving only finitely many negative powers of x_0 since λ satisfies the $P(z)$ -lower truncation condition. Thus the coefficients in x of $Y'_{P(z)}(v, x)\lambda$ satisfy the $P(z)$ -lower truncation condition.

By (5.86) and (5.87), we have

$$\begin{aligned}
& \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(u, x_1) \right) Y'_{P(z)}(v, x)\lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) (Y'_{P(z)}(v, x)\lambda)(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) (Y'_{P(z)}(v, x)\lambda)(w_{(1)} \otimes Y_2^o(u, x_1)w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \left(\lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \right. \\
& \quad \left. + \text{Res}_{x_2} z^{-1} \delta \left(\frac{x^{-1} - x_2}{z} \right) \right).
\end{aligned}$$

$$\begin{aligned}
& \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes w_{(2)}) \\
& + x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\left(\lambda(w_{(1)} \otimes Y_2^o(v, x)Y_2^o(u, x_1)w_{(2)})\right. \\
& \left. + \text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right)\lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)w_{(1)} \otimes Y_2^o(u, x_1)w_{(2)})\right).
\end{aligned} \tag{6.13}$$

Now the distributive law applies, giving us four terms. Then using the opposite commutator formula for Y_2^o (recall (2.61)) and the commutator formula for Y_2 , and (5.86), we can write the right-hand side of (6.13) as

$$\begin{aligned}
& z^{-1}\delta\left(\frac{x_1^{-1}-x_0}{z}\right)\lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\
& + z^{-1}\delta\left(\frac{x_1^{-1}-x_0}{z}\right)\text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)w_{(1)} \otimes w_{(2)}) \\
& + z^{-1}\delta\left(\frac{x_1^{-1}-x_0}{z}\right)\text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right)\text{Res}_{x_3}x_0^{-1}\delta\left(\frac{x_2-x_3}{x_0}\right) \cdot \\
& \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x_3)e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes w_{(2)}) \\
& + x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\lambda(w_{(1)} \otimes Y_2^o(u, x_1)Y_2^o(v, x)w_{(2)}) \\
& - x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\text{Res}_{x_3}x_1^{-1}\delta\left(\frac{x-x_3}{x_1}\right)\lambda(w_{(1)} \otimes Y_2^o(Y(v, x_3)u, x_1)w_{(2)}) \\
& + x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)w_{(1)} \otimes Y_2^o(u, x_1)w_{(2)}) \\
& = \left(\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(u, x_1)\right)\lambda\right)(w_{(1)} \otimes Y_2^o(v, x)w_{(2)}) \\
& + \text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right) \cdot \\
& \quad \cdot \left(\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(u, x_1)\right)\lambda\right)(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_2)w_{(1)} \otimes w_{(2)}) \\
& + \text{Res}_{x_2}z^{-1}\delta\left(\frac{x^{-1}-x_2}{z}\right)\text{Res}_{x_3}x_0^{-1}\delta\left(\frac{x_2-x_3}{x_0}\right)z^{-1}\delta\left(\frac{x_1^{-1}-x_0}{z}\right) \cdot \\
& \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x_3)e^{x_1L(1)}(-x_1^{-2})^{L(0)}u, x_0)w_{(1)} \otimes w_{(2)}) \\
& - \text{Res}_{x_3}x_1^{-1}\delta\left(\frac{x-x_3}{x_1}\right)x_0^{-1}\delta\left(\frac{z-x_1^{-1}}{-x_0}\right)\lambda(w_{(1)} \otimes Y_2^o(Y(v, x_3)u, x_1)w_{(2)}).
\end{aligned} \tag{6.14}$$

Since λ satisfies the $P(z)$ -compatibility condition (5.136), by (5.87) the sum of the first two terms of (6.14) is equal to

$$\begin{aligned} & \left(Y'_{P(z)}(v, x) \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(u, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) (Y'_{P(z)}(v, x) Y'_{P(z)}(u, x_1) \lambda) (w_{(1)} \otimes w_{(2)}). \end{aligned} \quad (6.15)$$

Changing the dummy variable x_3 to $-x_3 x^{-1} x_1^{-1}$ where we use x_3 to denote the new dummy variable, using (2.11), (2.6) and (6.1), and then evaluating Res_{x_2} , we see that the third term of (6.14) is equal to

$$\begin{aligned} & -\text{Res}_{x_2} \text{Res}_{x_3} z^{-1} \delta \left(\frac{x^{-1} - x_2}{z} \right) x^{-1} x_1^{-1} x_0^{-1} \delta \left(\frac{x_2 + x_3 x^{-1} x_1^{-1}}{x_0} \right) z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_2} z^{-1} \delta \left(\frac{x^{-1} - x_2}{z} \right) \text{Res}_{x_3} x^{-1} x_1^{-1} x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_3} \text{Res}_{x_2} z^{-1} \delta \left(\frac{x^{-1} - x_0 + x_3 x^{-1} x_1^{-1}}{z} \right) \cdot \\ & \quad \cdot x^{-1} x_1^{-1} x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_3} \text{Res}_{x_2} (z + x_0)^{-1} \delta \left(\frac{x^{-1} + x_3 x^{-1} x_1^{-1}}{z + x_0} \right) \cdot \\ & \quad \cdot x^{-1} x_1^{-1} x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) x_1 \delta \left(\frac{z + x_0}{x_1^{-1}} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_3} \text{Res}_{x_2} x_1 \delta \left(\frac{x^{-1} + x_3 x^{-1} x_1^{-1}}{x_1^{-1}} \right) \cdot \\ & \quad \cdot x^{-1} x_1^{-1} x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) x_1 \delta \left(\frac{z + x_0}{x_1^{-1}} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \\ &= -\text{Res}_{x_3} \text{Res}_{x_2} x_1^{-1} \delta \left(\frac{x - x_3}{x_1} \right) \cdot \\ & \quad \cdot x_2^{-1} \delta \left(\frac{x_0 - x_3 x^{-1} x_1^{-1}}{x_2} \right) z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \cdot \\ & \quad \cdot \lambda(Y_1(Y(e^{xL(1)}(-x^{-2})^{L(0)}v, -x_3 x^{-1} x_1^{-1}) e^{x_1 L(1)}(-x_1^{-2})^{L(0)}u, x_0) w_{(1)} \otimes w_{(2)}) \end{aligned}$$

$$\begin{aligned}
&= -\text{Res}_{x_3} \text{Res}_{x_2} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) x_2^{-1} \delta \left(\frac{x_0-x_3x_1^{-1}x_1^{-1}}{x_2} \right) z^{-1} \delta \left(\frac{x_1^{-1}-x_0}{z} \right) \\
&\quad \cdot \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}Y(v, x_3)u, x_0)w_{(1)} \otimes w_{(2)}) \\
&= -\text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) z^{-1} \delta \left(\frac{x_1^{-1}-x_0}{z} \right) \\
&\quad \cdot \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}Y(v, x_3)u, x_0)w_{(1)} \otimes w_{(2)}). \tag{6.16}
\end{aligned}$$

From (6.16), (5.86) and (5.136), the sum of the last two terms of (6.14) becomes

$$\begin{aligned}
&-\text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) z^{-1} \delta \left(\frac{x_1^{-1}-x_0}{z} \right) \\
&\quad \cdot \lambda(Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}Y(v, x_3)u, x_0)w_{(1)} \otimes w_{(2)}) \\
&\quad -\text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) x_0^{-1} \delta \left(\frac{z-x_1^{-1}}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2^o(Y(v, x_3)u, x_1)w_{(2)}) \\
&= -\text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) Y_t(Y(v, x_3)u, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
&= -x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) \text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) (Y'_{P(z)}(Y(v, x_3)u, x_1)\lambda) (w_{(1)} \otimes w_{(2)}). \tag{6.17}
\end{aligned}$$

Using (6.15), (6.17) and the commutator formula for $Y'_{P(z)}$, we now see that the right-hand side of (6.14) is equal to

$$\begin{aligned}
&x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) (Y'_{P(z)}(v, x)Y'_{P(z)}(u, x_1)\lambda) (w_{(1)} \otimes w_{(2)}) \\
&\quad -x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) \text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x-x_3}{x_1} \right) (Y'_{P(z)}(Y(v, x_3)u, x_1)\lambda) (w_{(1)} \otimes w_{(2)}) \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1}-z}{x_0} \right) (Y'_{P(z)}(u, x_1)Y'_{P(z)}(v, x)\lambda) (w_{(1)} \otimes w_{(2)}). \tag{6.18}
\end{aligned}$$

The formulas (6.13), (6.14) and (6.18) together prove (6.11), as desired. For the Möbius case, the corresponding verification for $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$ is straightforward, as usual, and we omit this verification. The first half of Theorem 5.40 holds.

For the second half of Theorem 5.40, suppose that $\lambda \in (W_1 \otimes W_2)^*$ satisfies either the $P(z)$ -local grading restriction condition or the $L(0)$ -semisimple condition. Assume without loss of generality that λ is doubly homogeneous. From Remark 5.37, we see that for $v \in V$ doubly homogeneous, $m \in \mathbb{Z}$ and $j = -1, 0, 1$, the elements $\tau_{P(z)}(v \otimes t^m)\lambda$ and $L'_{P(z)}(j)\lambda$ are also doubly homogeneous. Each such element μ lies in W_λ , and so $W_\mu \subset W_\lambda$. Thus μ satisfies the $P(z)$ -local grading restriction condition (or the $L(0)$ -semisimple condition), and the second half of Theorem 5.40 follows. \square

6.2 Proofs of Theorems 5.70 and 5.71

In this subsection, we follow [HL6]; the arguments given there carry over to our more general situation with very little change. We first prove certain formulas that will be useful later.

Let $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. From (5.169) we have

$$\begin{aligned} & (L'_{Q(z)}(0)\lambda)(w_{(1)} \otimes w_{(2)}) \\ &= \lambda((L(0) - zL(1))w_{(1)} \otimes w_{(2)}) - \lambda(w_{(1)} \otimes (L(0) - zL(-1))w_{(2)}), \end{aligned} \quad (6.19)$$

where (as usual) we have used the same notations $L(0), L(-1), L(1)$ to denote operators on both W_1 and W_2 . For convenience we write $L(-1) = L'_{Q(z)}(-1)$, $L(0) = L'_{Q(z)}(0)$ and $L(1) = L'_{Q(z)}(1)$ in the rest of this section. There will be no confusion since the operators act on different spaces.

Lemma 6.2 *For $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, we have*

$$\begin{aligned} & \left(\left(1 - \frac{y_1}{z}\right)^{L(0)} \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= \lambda \left(\left(1 - \frac{y_1}{z}\right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(-1))} w_{(2)} \right). \end{aligned} \quad (6.20)$$

Proof From (6.19),

$$\begin{aligned} & \left(\left(1 - \frac{y_1}{z}\right)^{L(0)} \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= (e^{L(0) \log(1 - \frac{y_1}{z})} \lambda) (w_{(1)} \otimes w_{(2)}) \\ &= \lambda(e^{(L(0)-zL(1)) \log(1 - \frac{y_1}{z})} w_{(1)} \otimes e^{-(L(0)-zL(-1)) \log(1 - \frac{y_1}{z})} w_{(2)}) \\ &= \lambda \left(\left(1 - \frac{y_1}{z}\right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z}\right)^{-(L(0)-zL(-1))} w_{(2)} \right). \quad \square \end{aligned} \quad (6.21)$$

Lemma 6.3 *For $v \in V$,*

$$Y'_{Q(z)}(v, x) = \left(1 - \frac{y_1}{z}\right)^{L(0)} Y'_{Q(z)} \left(\left(1 - \frac{y_1}{z}\right)^{-L(0)} v, \frac{x}{1 - y_1/z} \right) \left(1 - \frac{y_1}{z}\right)^{-L(0)}. \quad (6.22)$$

This formula also holds for the vertex operators associated with any generalized V -module.

Proof The identity (6.22) will follow from the formula

$$e^{yL(0)} Y'_{Q(z)}(v, x) e^{-yL(0)} = Y'_{Q(z)}(e^{yL(0)} v, e^y x). \quad (6.23)$$

To prove this, assume without loss of generality that $\text{wt } v = h \in \mathbb{Z}$, and use the $L(-1)$ -derivative property (5.154) and the commutator formulas (5.160) and (5.172) to get

$$[L(0), Y'_{Q(z)}(v, x)] = \left(x \frac{d}{dx} + h \right) Y'_{Q(z)}(v, x). \quad (6.24)$$

Formula (6.23) now follows from (an easier version of) the proof of (3.84). \square

Lemma 6.4 *Let $L(-1)$ and $L(0)$ be any operators satisfying the commutator relation*

$$[L(0), L(-1)] = L(-1). \quad (6.25)$$

Then

$$\left(1 - \frac{y_1}{x}\right)^{L(0)-xL(-1)} = e^{y_1 L(-1)} \left(1 - \frac{y_1}{x}\right)^{L(0)}. \quad (6.26)$$

Proof We first prove that the derivative with respect to y of

$$(1-y)^{L(0)-xL(-1)}(1-y)^{-L(0)}e^{-xyL(-1)}$$

is 0. Write $A = (1-y)^{L(0)-xL(-1)}$, $B = (1-y)^{-L(0)}$, $C = e^{-xyL(-1)}$. Then

$$\begin{aligned} \frac{d}{dy}(ABC) &= -A(1-y)^{-1}(L(0) - xL(-1))BC \\ &\quad + A(1-y)^{-1}L(0)BC \\ &\quad - xABL(-1)C. \end{aligned} \quad (6.27)$$

Using (3.71) we have

$$\begin{aligned} BL(-1) &= (1-y)^{-L(0)}L(-1) \\ &= e^{(-\log(1-y))L(0)}L(-1) \\ &= L(-1)e^{(-\log(1-y))L(0)}e^{-\log(1-y)} \\ &= L(-1)(1-y)^{-L(0)}(1-y)^{-1} \\ &= (1-y)^{-1}L(-1)B, \end{aligned} \quad (6.28)$$

and substituting (6.28) into (6.27) gives

$$\begin{aligned} \frac{d}{dy}(ABC) &= -A(1-y)^{-1}L(0)BC + xA(1-y)^{-1}L(-1)BC \\ &\quad + A(1-y)^{-1}L(0)BC - xA(1-y)^{-1}L(-1)BC \\ &= 0. \end{aligned}$$

Thus ABC is constant in y , and since $ABC|_{y=0} = 1$, we have $ABC = 1$, which is equivalent to (6.26). \square

Proof of Theorem 5.70 As always, the reader should again observe the justifiability of each formal step in the argument.

Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $Q(z)$ -compatibility condition, that is, (a) the $Q(z)$ -lower truncation condition—for all $v \in V$, $Y'_{Q(z)}(v, x)\lambda = \tau_{Q(z)}(Y_t(v, x))\lambda$ involves only finitely many negative powers of x , and (b)

$$\begin{aligned} \tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right)Y_t(v, x_0)\right)\lambda \\ = z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right)Y'_{Q(z)}(v, x_0)\lambda \quad \text{for all } v \in V. \end{aligned} \quad (6.29)$$

By (5.151) and (5.152), (6.29) is equivalent to

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \lambda(Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)}) \\
& - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \left(\text{Res}_{y_1} x_0^{-1} \delta \left(\frac{y_1 - z}{x_0} \right) \lambda(Y_1^o(v, y_1)w_{(1)} \otimes w_{(2)}) \right. \\
& \quad \left. - \text{Res}_{y_1} x_0^{-1} \delta \left(\frac{z - y_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, y_1)w_{(2)}) \right)
\end{aligned} \tag{6.30}$$

for all $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. It is important to note that on the right-hand side the distributive law is not valid since the two individual products are not defined. One critical feature of the argument that follows is that we must rewrite expressions to allow the application of distributivity.

By (5.152), we have

$$\begin{aligned}
& \left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{Q(z)}(v_1, x_1) Y'_{Q(z)}(v_2, x_2) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (Y'_{Q(z)}(v_1, x_1) Y'_{Q(z)}(v_2, x_2) \lambda) (w_{(1)} \otimes w_{(2)}) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) \cdot \right. \\
& \quad \cdot (Y'_{Q(z)}(v_2, x_2) \lambda) (Y_1^o(v_1, y_1)w_{(1)} \otimes w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2) \lambda) (w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{y_2 - z}{x_2} \right) \cdot \right. \\
& \quad \cdot \lambda(Y_1^o(v_2, y_2)Y_1^o(v_1, y_1)w_{(1)} \otimes w_{(2)}) \\
& \quad - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{y_1 - z}{x_1} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(v_1, y_1)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2) \lambda) (w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right).
\end{aligned} \tag{6.31}$$

From the properties of the formal δ -function, we see that the right-hand side of (6.31) is equal to

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\text{Res}_{y_1} y_1^{-1} \delta \left(\frac{x_1 + z}{y_1} \right) \text{Res}_{y_2} y_2^{-1} \delta \left(\frac{x_2 + z}{y_2} \right) \cdot \right. \\
& \quad \cdot \lambda(Y_1^o(v_2, x_2 + z)Y_1^o(v_1, x_1 + z)w_{(1)} \otimes w_{(2)})
\end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{y_1} y_1^{-1} \delta \left(\frac{x_1 + z}{y_1} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(v_1, x_1 + z)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& -\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\lambda(Y_1^o(v_2, x_2 + z)Y_1^o(v_1, x_1 + z)w_{(1)} \otimes w_{(2)}) \right. \\
& \quad - \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(Y_1^o(v_1, x_1 + z)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right). \tag{6.32}
\end{aligned}$$

From the $L(-1)$ -derivative property for Y_1 , the $L(-1)$ -derivative property (2.62) for Y_1^o and the commutator formulas for $L(-1)$, $Y_1(\cdot, x)$ and for $L(1)$, $Y_1^o(\cdot, x)$ (recall Lemma 2.22), we obtain

$$\begin{aligned}
Y_1(v, x + z) &= Y_1(e^{zL(-1)}v, x) \\
&= e^{zL(-1)}Y_1(v, x)e^{-zL(-1)} \\
&= \sum_{n \geq 0} \frac{z^n}{n!} \frac{d^n}{dx^n} Y_1(v, x) \tag{6.33}
\end{aligned}$$

and

$$\begin{aligned}
Y_1^o(v, x + z) &= Y_1^o(e^{zL(-1)}v, x) \\
&= e^{-zL(1)}Y_1^o(v, x)e^{zL(1)} \\
&= \sum_{n \geq 0} \frac{z^n}{n!} \frac{d^n}{dx^n} Y_1^o(v, x). \tag{6.34}
\end{aligned}$$

(Note that all these expressions are in fact defined.) Using (6.34), we see that the right-hand side of (6.32) can be written as

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\lambda(Y_1^o(e^{zL(-1)}v_2, x_2)Y_1^o(e^{zL(-1)}v_1, x_1)w_{(1)} \otimes w_{(2)}) \right. \\
& \quad - \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(Y_1^o(e^{zL(-1)}v_1, x_1)w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \left(\lambda(e^{-zL(1)}Y_1^o(v_2, x_2)Y_1^o(v_1, x_1)e^{zL(1)}w_{(1)} \otimes w_{(2)}) \right. \\
& \quad - \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(e^{-zL(1)}Y_1^o(v_1, x_1)e^{zL(1)}w_{(1)} \otimes Y_2(v_2, y_2)w_{(2)}) \\
& \quad \left. - \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2)\lambda)(w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \right). \tag{6.35}
\end{aligned}$$

Note that it is easier to verify the well-definedness of the terms on the right-hand side of (6.32) than that of the terms in (6.35), though (6.35) is sometimes easier to use if it is known that every term is well defined. Below we shall write expressions like those on the right-hand side of (6.32) in whichever way suits our needs. The distributive law applies to the right-hand side of (6.32) (or (6.35)) since all three of the following expressions are defined:

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \lambda(Y_1^o(v_2, x_2 + z) Y_1^o(v_1, x_1 + z) w_{(1)} \otimes w_{(2)}), \\ & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \lambda(Y_1^o(v_1, x_1 + z) w_{(1)} \otimes Y_2(v_2, y_2) w_{(2)}), \\ & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2) \lambda)(w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}). \end{aligned}$$

Now we examine the last expression in (6.35). Rewriting the formal δ -functions $x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right)$ and $x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right)$, and using Lemma 6.3 and (2.11), we have:

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot (Y'_{Q(z)}(v_2, x_2) \lambda)(w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}) \\ &= \left(\frac{x_1}{z} \right)^{-1} \left(\frac{x_0}{x_1/z} \right)^{-1} \delta \left(\frac{z + \left(\frac{x_2}{-x_1/z} \right)}{\frac{x_0}{x_1/z}} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \cdot \\ & \quad \cdot \left(\left(1 - \frac{y_1}{z} \right)^{L(0)} Y'_{Q(z)} \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_2}{1 - y_1/z} \right) \cdot \right. \\ & \quad \cdot \left. \left(1 - \frac{y_1}{z} \right)^{-L(0)} \lambda \right) (w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}) \\ &= \left(\frac{x_1}{z} \right)^{-1} \left(\frac{x_0}{x_1/z} \right)^{-1} \delta \left(\frac{z + \left(\frac{x_2}{-x_1/z} \right)}{\frac{x_0}{x_1/z}} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \cdot \\ & \quad \cdot \left(\left(1 - \frac{y_1}{z} \right)^{L(0)} Y'_{Q(z)} \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_2}{-x_1/z} \right) \cdot \right. \\ & \quad \cdot \left. \left(1 - \frac{y_1}{z} \right)^{-L(0)} \lambda \right) (w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}). \end{aligned} \tag{6.36}$$

By Lemma 6.2 and (6.29), the right-hand side of (6.36) is equal to

$$\begin{aligned} & \left(\frac{x_1}{z} \right)^{-1} \left(\frac{x_0}{x_1/z} \right)^{-1} \delta \left(\frac{z + \left(\frac{x_2}{-x_1/z} \right)}{\frac{x_0}{x_1/z}} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \cdot \\ & \quad \cdot \left(Y'_{Q(z)} \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_2}{-x_1/z} \right) \left(1 - \frac{y_1}{z} \right)^{-L(0)} \lambda \right). \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\left(1 - \frac{y_1}{z} \right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \right) \\
& = \left(\frac{x_1}{z} \right)^{-1} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \cdot \\
& \quad \cdot \left(\tau_{Q(z)} \left(\left(\frac{x_0}{x_1/z} \right)^{-1} \delta \left(\frac{z + (\frac{x_2}{-x_1/z})}{\frac{x_0}{x_1/z}} \right) \right) \cdot \right. \\
& \quad \cdot Y_t \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_2}{-x_1/z} \right) \left(1 - \frac{y_1}{z} \right)^{-L(0)} \lambda \Big) \cdot \\
& \quad \cdot \left(\left(1 - \frac{y_1}{z} \right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \right). \quad (6.37)
\end{aligned}$$

By (5.151), the right-hand side of (6.37) becomes

$$\begin{aligned}
& \left(\frac{x_1}{z} \right)^{-1} \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \left(\left(\frac{x_2}{-x_1/z} \right)^{-1} \delta \left(\frac{\frac{x_0}{x_1/z} - z}{\frac{x_2}{-x_1/z}} \right) \cdot \right. \\
& \quad \cdot \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} \lambda \right) \left(Y_1^o \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{x_1/z} \right) \cdot \right. \\
& \quad \cdot \left(1 - \frac{y_1}{z} \right)^{L(0)-zL(1)} w_{(1)} \otimes \left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(1))} Y_2(v_1, y_1) w_{(2)} \Big) \\
& \quad - \left(\frac{x_2}{-x_1/z} \right)^{-1} \delta \left(\frac{z - \frac{x_0}{x_1/z}}{-\frac{x_2}{-x_1/z}} \right) \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} \lambda \right) \left(\left(1 - \frac{y_1}{z} \right)^{L(0)-zL(1)} \cdot \right. \\
& \quad \cdot w_{(1)} \otimes Y_2 \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{x_1/z} \right) \cdot \\
& \quad \cdot \left. \left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \right) \Big). \quad (6.38)
\end{aligned}$$

Using Lemma 6.2 again but with $1 - \frac{y_1}{z}$ replaced by $(1 - \frac{y_1}{z})^{-1}$, rewriting formal δ -functions and then using the distributive law, we see that (6.38) is equal to

$$\begin{aligned}
& \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) \left(-x_2^{-1} \delta \left(\frac{x_0 - x_1}{-x_2} \right) \cdot \right. \\
& \quad \cdot \lambda \left(\left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(1))} Y_1^o \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \cdot \right. \\
& \quad \cdot \left(1 - \frac{y_1}{z} \right)^{L(0)-zL(1)} w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)} \Big) \\
& \quad + x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda \left(w_{(1)} \otimes \left(1 - \frac{y_1}{z} \right)^{L(0)-zL(-1)} \cdot \right. \\
& \quad \cdot Y_2 \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \cdot \\
& \quad \cdot \left. \left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \right) \Big)
\end{aligned}$$

$$\begin{aligned}
&= -\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) x_2^{-1} \delta \left(\frac{x_0 - x_1}{-x_2} \right) \cdot \\
&\quad \cdot \lambda \left(\left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(1))} Y_1^o \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \right) \cdot \\
&\quad \cdot \left(1 - \frac{y_1}{z} \right)^{L(0)-zL(1)} w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)} \Big) \\
&+ \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{1 - y_1/z}{-x_1/z} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \lambda \left(w_{(1)} \otimes \right. \\
&\quad \otimes \left(1 - \frac{y_1}{z} \right)^{L(0)-zL(-1)} Y_2 \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \Big) \cdot \\
&\quad \cdot \left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(-1))} Y_2(v_1, y_1) w_{(2)} \Big). \tag{6.39}
\end{aligned}$$

But by Lemmas 6.4 and 6.3,

$$\begin{aligned}
&\left(1 - \frac{y_1}{z} \right)^{L(0)-zL(-1)} Y_2 \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(-1))} \\
&= e^{y_1 L(-1)} \left(1 - \frac{y_1}{z} \right)^{L(0)} Y_2 \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \cdot \\
&\quad \cdot \left(1 - \frac{y_1}{z} \right)^{-L(0)} e^{-y_1 L(-1)} \\
&= e^{y_1 L(-1)} Y_2(v_2, -x_0) e^{-y_1 L(-1)} \\
&= Y_2(v_2, -x_0 + y_1) \\
&= Y_2(v_2, -x_0 - (z - y_1) + z) \\
&= Y_2(e^{zL(-1)} v_2, -x_0 - (z - y_1)). \tag{6.40}
\end{aligned}$$

We similarly have, using Lemmas 6.4 and 6.3 for $Y_1'(v_2, x)$ and then using (2.73) and Theorem 2.34,

$$\begin{aligned}
&\left(1 - \frac{y_1}{z} \right)^{-(L(0)-zL(1))} Y_1^o \left(\left(1 - \frac{y_1}{z} \right)^{-L(0)} v_2, \frac{x_0}{-(1 - y_1/z)} \right) \left(1 - \frac{y_1}{z} \right)^{L(0)-zL(1)} \\
&= e^{-y_1 L(1)} Y_1^o(v_2, -x_0) e^{y_1 L(1)} \\
&= Y_1^o(v_2, -x_0 + y_1) \\
&= Y_1^o(v_2, -x_0 - (z - y_1) + z) \\
&= Y_1^o(e^{zL(-1)} v_2, -x_0 - (z - y_1)). \tag{6.41}
\end{aligned}$$

Substituting (6.40) and (6.41) into the right-hand side of (6.39) and then combining with (6.36)–(6.39), we obtain

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) (Y'_{Q(z)}(v_2, x_2) \lambda)(w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)})$$

$$\begin{aligned}
&= -\text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{x_0 - x_1}{-x_2} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_2, -x_0 - (z - y_1) + z)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&+ \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, -x_0 + y_1)Y_2(v_1, y_1)w_{(2)}). \tag{6.42}
\end{aligned}$$

(We choose the form of the expression from (6.41) in anticipation of the next step.)

By (2.11) and (6.33), the right-hand side of (6.42) is equal to

$$\begin{aligned}
&-x_2^{-1} \delta \left(\frac{-x_0 + x_1}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(v_2, -x_0 + x_1 + z)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&+ x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, -x_0 + y_1)Y_2(v_1, y_1)w_{(2)}) \\
&= -x_2^{-1} \delta \left(\frac{-x_0 + x_1}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&+ x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, -x_0 + y_1)Y_2(v_1, y_1)w_{(2)}). \tag{6.43}
\end{aligned}$$

Since

$$\text{Res}_{y_2} y_2^{-1} \delta \left(\frac{-x_0 + y_1}{y_2} \right) = 1,$$

the right-hand side of (6.43) can be written as

$$\begin{aligned}
&-x_2^{-1} \delta \left(\frac{-x_0 + x_1}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
&+ x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \text{Res}_{y_2} y_2^{-1} \delta \left(\frac{-x_0 + y_1}{y_2} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, -x_0 + y_1)Y_2(v_1, y_1)w_{(2)}). \tag{6.44}
\end{aligned}$$

By (2.6) and (2.11), (6.44) becomes

$$\begin{aligned}
&x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
&\quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)})
\end{aligned}$$

$$\begin{aligned}
& +x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right)\text{Res}_{y_2}y_2^{-1}\delta\left(\frac{-x_0+y_1}{y_2}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}) \\
& = x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& \quad + x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right)y_2^{-1}\delta\left(\frac{-x_0+y_1}{y_2}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}) \\
& = x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& \quad - x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2} \cdot \\
& \quad \cdot (x_2+x_0)^{-1}\delta\left(\frac{z-y_1}{-x_2-x_0}\right)x_0^{-1}\delta\left(\frac{y_2-y_1}{-x_0}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}) \\
& = x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& \quad - x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right)x_0^{-1}\delta\left(\frac{y_2-y_1}{-x_0}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}). \tag{6.45}
\end{aligned}$$

Substituting (6.42)–(6.45) into (6.35) we obtain

$$\begin{aligned}
& \left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y'_{Q(z)}(v_1, x_1)Y'_{Q(z)}(v_2, x_2)\lambda\right)(w_{(1)} \otimes w_{(2)}) \\
& = x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\lambda(Y_1^o(e^{zL(-1)}v_2, x_2)Y_1^o(e^{zL(-1)}v_1, x_1)w_{(1)} \otimes w_{(2)}) \\
& \quad - x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_1, x_1)w_{(1)} \otimes Y_2(v_2, x_2)w_{(2)}) \\
& \quad - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\text{Res}_{y_1}x_1^{-1}\delta\left(\frac{z-y_1}{-x_1}\right) \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)}v_2, x_2)w_{(1)} \otimes Y_2(v_1, y_1)w_{(2)}) \\
& \quad + x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right)x_0^{-1}\delta\left(\frac{y_2-y_1}{-x_0}\right) \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_2, y_2)Y_2(v_1, y_1)w_{(2)}). \tag{6.46}
\end{aligned}$$

Now consider the result of the calculation from (6.42) to (6.45) except for the last two steps in (6.45). Reversing the subscripts 1 and 2 of the symbols v , x and y and replacing x_0 by $-x_0$ in this result and then using (2.6), we have

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot (Y'_{Q(z)}(v_1, x_1) \lambda)(w_{(1)} \otimes Y_2(v_2, y_2) w_{(2)}) \\
& = x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)} v_1, x_1) w_{(1)} \otimes Y_2(v_2, x_2) w_{(2)}) \\
& \quad - x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) x_0^{-1} \delta \left(\frac{y_1 - y_2}{x_0} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_1, y_1) Y_2(v_2, y_2) w_{(2)}). \tag{6.47}
\end{aligned}$$

From (6.31)–(6.35), again reversing the subscripts 1 and 2 of the symbols v , x and y and replacing x_0 by $-x_0$, and (6.47), we have

$$\begin{aligned}
& \left(-x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{Q(z)}(v_2, x_2) Y'_{Q(z)}(v_1, x_1) \cdot \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = -x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \lambda(Y_1^o(e^{zL(-1)} v_1, x_1) Y_1^o(e^{zL(-1)} v_2, x_2) w_{(1)} \otimes w_{(2)}) \\
& \quad + x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \text{Res}_{y_1} x_1^{-1} \delta \left(\frac{z - y_1}{-x_1} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)} v_2, x_2) w_{(1)} \otimes Y_2(v_1, y_1) w_{(2)}) \\
& \quad + x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(Y_1^o(e^{zL(-1)} v_1, x_1) w_{(1)} \otimes Y_2(v_2, x_2) w_{(2)}) \\
& \quad - x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \text{Res}_{y_1} \text{Res}_{y_2} x_2^{-1} \delta \left(\frac{z - y_2}{-x_2} \right) x_0^{-1} \delta \left(\frac{y_1 - y_2}{x_0} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2(v_1, y_1) Y_2(v_2, y_2) w_{(2)}). \tag{6.48}
\end{aligned}$$

The formulas (6.46) and (6.48) give:

$$\begin{aligned}
& \left(\left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{Q(z)}(v_1, x_1) Y'_{Q(z)}(v_2, x_2) \right. \right. \\
& \quad \left. \left. - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{Q(z)}(v_2, x_2) Y'_{Q(z)}(v_1, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = \lambda \left(\left(x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_1^o(e^{zL(-1)} v_2, x_2) Y_1^o(e^{zL(-1)} v_1, x_1) \right. \right. \\
& \quad \left. \left. - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_1^o(e^{zL(-1)} v_1, x_1) Y_1^o(e^{zL(-1)} v_2, x_2) \right) w_{(1)} \otimes w_{(2)} \right)
\end{aligned}$$

$$\begin{aligned}
& -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right) \\
& \quad \cdot \lambda\left(w_{(1)}\otimes\left(x_0^{-1}\delta\left(\frac{y_1-y_2}{x_0}\right)Y_2(v_1,y_1)Y_2(v_2,y_2)\right.\right. \\
& \quad \left.\left.-x_0^{-1}\delta\left(\frac{y_2-y_1}{-x_0}\right)Y_2(v_2,y_2)Y_2(v_1,y_1)\right)w_{(2)}\right). \tag{6.49}
\end{aligned}$$

From the Jacobi identities for Y_1^o and Y_2 and (3.58), the right-hand side of (6.49) is equal to

$$\begin{aligned}
& x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(Y_1^o(Y(e^{zL(-1)}v_1,x_0)e^{zL(-1)}v_2,x_2)w_{(1)}\otimes w_{(2)}) \\
& \quad -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right)y_2^{-1}\delta\left(\frac{y_1-x_0}{y_2}\right) \\
& \quad \cdot \lambda(w_{(1)}\otimes Y_2(Y(v_1,x_0)v_2,y_2)w_{(2)}) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(Y_1^o(e^{zL(-1)}Y(v_1,x_0)v_2,x_2)w_{(1)}\otimes w_{(2)}) \\
& \quad -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_1}\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right)y_2^{-1}\delta\left(\frac{y_1-x_0}{y_2}\right) \\
& \quad \cdot \lambda(w_{(1)}\otimes Y_2(Y(v_1,x_0)v_2,y_2)w_{(2)}). \tag{6.50}
\end{aligned}$$

Using (6.34), evaluating Res_{y_1} and then using the definition of $Y'_{Q(z)}$ (recall (5.152)), we finally see that the right-hand side of (6.50) is equal to

$$\begin{aligned}
& x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\lambda(Y_1^o(Y(v_1,x_0)v_2,x_2+z)w_{(1)}\otimes w_{(2)}) \\
& \quad -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\text{Res}_{y_2}x_2^{-1}\delta\left(\frac{z-y_2}{-x_2}\right) \\
& \quad \cdot \lambda(w_{(1)}\otimes Y_2(Y(v_1,x_0)v_2,y_2)w_{(2)}) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)(Y'_{Q(z)}(Y(v_1,x_0)v_2,x_2)\lambda)(w_{(1)}\otimes w_{(2)}), \tag{6.51}
\end{aligned}$$

proving Theorem 5.70. \square

Proof of Theorem 5.71 Let λ be an element of $(W_1\otimes W_2)^*$ satisfying the $Q(z)$ -compatibility condition. We first want to prove that each coefficient in x of $Y'_{Q(z)}(u,x_0)Y'_{Q(z)}(v,x)\lambda$ is a formal Laurent series involving only finitely many negative powers of x_0 and that

$$\begin{aligned}
& \tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(u,x_0)\right)Y'_{Q(z)}(v,x)\lambda \\
& = z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y'_{Q(z)}(u,x_0)Y'_{Q(z)}(v,x)\lambda \tag{6.52}
\end{aligned}$$

for all $u, v \in V$. Using the commutator formula for $Y'_{Q(z)}$, we have

$$\begin{aligned} & Y'_{Q(z)}(u, x_0)Y'_{Q(z)}(v, x)\lambda \\ &= Y'_{Q(z)}(v, x)Y'_{Q(z)}(u, x_0)\lambda \\ & \quad - \text{Res}_y x_0^{-1} \delta \left(\frac{x-y}{x_0} \right) Y'_{Q(z)}(Y(v, y)u, x_0)\lambda. \end{aligned} \quad (6.53)$$

Each coefficient in x of the right-hand side of (6.53) is a formal Laurent series involving only finitely many negative powers of x_0 since λ satisfies the $Q(z)$ -lower truncation condition. Thus the coefficients in x of $Y'_{Q(z)}(v, x)\lambda$ satisfy the $Q(z)$ -lower truncation condition.

By (5.151) and (5.152), we have

$$\begin{aligned} & \left(\tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(u, x_0) \right) Y'_{Q(z)}(v, x)\lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) (Y'_{Q(z)}(v, x)\lambda) (Y_1^o(u, x_1)w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) (Y'_{Q(z)}(v, x)\lambda) (w_{(1)} \otimes Y_2(u, x_1)w_{(2)}) \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \left(\lambda(Y_1^o(v, x+z)Y_1^o(u, x_1)w_{(1)} \otimes w_{(2)}) \right. \\ & \quad \left. - \text{Res}_{x_2} x^{-1} \delta \left(\frac{z - x_2}{-x} \right) \lambda(Y_1^o(u, x_1)w_{(1)} \otimes Y_2(v, x_2)w_{(2)}) \right) \\ & \quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \left(\lambda(Y_1^o(e^{zL(-1)}v, x)w_{(1)} \otimes Y_2(u, x_1)w_{(2)}) \right. \\ & \quad \left. - \text{Res}_{x_2} x^{-1} \delta \left(\frac{z - x_2}{-x} \right) \lambda(w_{(1)} \otimes Y_2(v, x_2)Y_2(u, x_1)w_{(2)}) \right). \end{aligned} \quad (6.54)$$

Now the distributive law applies, giving us four terms. Inserting

$$\text{Res}_{x_4} x_4^{-1} \delta \left(\frac{x+z}{x_4} \right) = 1$$

into the first of these terms and correspondingly replacing $x+z$ by x_4 in $Y_1^o(v, x+z)$, we can apply the commutator formula for Y_1^o in the usual way. Also using the commutator formula for Y_2 , (2.6) and (2.11), we write the right-hand side of (6.54) as

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \lambda(Y_1^o(u, x_1)Y_1^o(v, x+z)w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \text{Res}_{x_4} \text{Res}_{x_3} x_1^{-1} \delta \left(\frac{x_4 - x_3}{x_1} \right) x_4^{-1} \delta \left(\frac{x+z}{x_4} \right) \\ & \quad \quad \cdot \lambda(Y_1^o(Y(v, x_3)u, x_1)w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \text{Res}_{x_2} x^{-1} \delta \left(\frac{z - x_2}{-x} \right) \lambda(Y_1^o(u, x_1)w_{(1)} \otimes Y_2(v, x_2)w_{(2)}) \end{aligned}$$

$$\begin{aligned}
& -x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\lambda(Y_1^o(e^{zL(-1)}v,x)w_{(1)}\otimes Y_2(u,x_1)w_{(2)}) \\
& +x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\text{Res}_{x_2}x^{-1}\delta\left(\frac{z-x_2}{-x}\right)\lambda(w_{(1)}\otimes Y_2(u,x_1)Y_2(v,x_2)w_{(2)}) \\
& +x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\text{Res}_{x_2}x^{-1}\delta\left(\frac{z-x_2}{-x}\right)\text{Res}_{x_3}x_1^{-1}\delta\left(\frac{x_2-x_3}{x_1}\right) \\
& \quad \cdot\lambda(w_{(1)}\otimes Y_2(Y(v,x_3)u,x_1)w_{(2)}) \\
& = \left(x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\lambda(Y_1^o(u,x_1)Y_1^o(e^{zL(-1)}v,x)w_{(1)}\otimes w_{(2)})\right. \\
& \quad \left.-x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\lambda(Y_1^o(e^{zL(-1)}v,x)w_{(1)}\otimes Y_2(u,x_1)w_{(2)})\right) \\
& \quad -\text{Res}_{x_2}x^{-1}\delta\left(\frac{z-x_2}{-x}\right)\left(x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\right. \\
& \quad \quad \cdot\lambda(Y_1^o(u,x_1)w_{(1)}\otimes Y_2(v,x_2)w_{(2)}) \\
& \quad \quad \left.-x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\lambda(w_{(1)}\otimes Y_2(u,x_1)Y_2(v,x_2)w_{(2)})\right) \\
& \quad -\left(\text{Res}_{x_4}\text{Res}_{x_3}(x_0+z)^{-1}\delta\left(\frac{x+z-x_3}{x_0+z}\right)\right. \\
& \quad \quad \cdot x_4^{-1}\delta\left(\frac{x+z}{x_4}\right)x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right) \\
& \quad \quad \cdot\lambda(Y_1^o(Y(v,x_3)u,x_1)w_{(1)}\otimes w_{(2)}) \\
& \quad \quad \left.-\text{Res}_{x_2}\text{Res}_{x_3}x^{-1}\delta\left(\frac{z-(x_1+x_3)}{-x}\right)x_2^{-1}\delta\left(\frac{x_1+x_3}{x_2}\right)\right. \\
& \quad \quad \left.\cdot x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\lambda(w_{(1)}\otimes Y_2(Y(v,x_3)u,x_1)w_{(2)})\right). \tag{6.55}
\end{aligned}$$

Using (5.151) and (2.6) and evaluating suitable residues, we see that the right-hand side of (6.55) is equal to

$$\begin{aligned}
& \left(\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(u,x_0)\right)\lambda\right)(Y_1^o(e^{zL(-1)}v,x)w_{(1)}\otimes w_{(2)}) \\
& -\text{Res}_{x_2}x^{-1}\delta\left(\frac{z-x_2}{-x}\right) \\
& \quad \cdot\left(\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(u,x_0)\right)\lambda\right)(w_{(1)}\otimes Y_2(v,x_2)w_{(2)}) \\
& -\left(\text{Res}_{x_3}x_0^{-1}\delta\left(\frac{x-x_3}{x_0}\right)\right. \\
& \quad \cdot x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\lambda(Y_1^o(Y(v,x_3)u,x_1)w_{(1)}\otimes w_{(2)})
\end{aligned}$$

$$\begin{aligned}
& -\text{Res}_{x_3} x^{-1} \delta \left(\frac{x_0 + x_3}{x} \right) \cdot \\
& \cdot x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(Y(v, x_3)u, x_1)w_{(2)})) \Big). \tag{6.56}
\end{aligned}$$

Using (5.151) and (5.152), we find that the right-hand side of (6.56) becomes

$$\begin{aligned}
& \left(Y'_{Q(z)}(v, x) \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(u, x_0) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& - \left(\text{Res}_{x_3} x_0^{-1} \delta \left(\frac{x - x_3}{x_0} \right) \cdot \right. \\
& \cdot \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(Y(v, x_3)u, x_0) \right) \lambda \Big) (w_{(1)} \otimes w_{(2)}). \tag{6.57}
\end{aligned}$$

By the compatibility condition for λ and the commutator formula for $Y'_{Q(z)}$, the right-hand side of (6.57) is equal to

$$\begin{aligned}
& z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) (Y'_{Q(z)}(v, x) Y'_{Q(z)}(u, x_0) \lambda) (w_{(1)} \otimes w_{(2)}) \\
& - z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \left(\text{Res}_{x_3} x_0^{-1} \delta \left(\frac{x - x_3}{x_0} \right) \cdot \right. \\
& \quad \cdot Y'_{Q(z)}(Y(v, x_3)u, x_0) \lambda \Big) (w_{(1)} \otimes w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) \left(\left(Y'_{Q(z)}(v, x) Y'_{Q(z)}(u, x_0) \right. \right. \\
& \quad \left. \left. - \text{Res}_{x_3} x_0^{-1} \delta \left(\frac{x - x_3}{x_0} \right) Y'_{Q(z)}(Y(v, x_3)u, x_0) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) (Y'_{Q(z)}(u, x_0) Y'_{Q(z)}(v, x) \lambda) (w_{(1)} \otimes w_{(2)}). \tag{6.58}
\end{aligned}$$

This proves (6.52). In the Möbius case, the three operators are handled in the usual way. The first part of Theorem 5.71 is established.

The proof of the second half of Theorem 5.71 is exactly like that for Theorem 5.40. \square

References

- [A1] D. Adamović, Rationality of Neveu-Schwarz vertex operator superalgebras, *Internat. Math. Res. Notices* **1997** (1997), 865–874.
- [A2] D. Adamović, Representations of the $N = 2$ superconformal vertex algebra, *Internat. Math. Res. Notices* **1999** (1999), 61–79.
- [A3] D. Adamović, Rationality of unitary $N = 2$ vertex operator superalgebras, to appear; math.QA/9909055.
- [BK] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*, University Lecture Series, Vol. 21, Amer. Math. Soc., Providence, RI, 2001.
- [BFM] A. Beilinson, B. Feigin and B. Mazur, Introduction to algebraic field theory on curves, preprint, 1991 (provided by A. Beilinson, 1996).
- [BPZ] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory, *Nuclear Phys. B* **241** (1984), 333–380.
- [B] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.
- [CF] N. Carqueville and M. Flohr, Nonmeromorphic operator product expansion and C_2 -cofiniteness for a family of \mathcal{W} -algebras, *J. Phys.* **A39** (2006), 951–966.
- [De] P. Deligne, Une description de catégorie tressée (inspiré par Drinfeld), unpublished.
- [D1] C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **160** (1993), 245–265.
- [D2] C. Dong, Representations of the moonshine module vertex operator algebra, in: *Mathematical aspects of conformal and topological field theories and quantum groups*, South Hadley, MA, 1992, Contemporary Math., Vol. 175, Amer. Math. Soc., Providence, RI, 1994, 27–36.
- [DL] C. Dong and J. Lepowsky, *Generalized Vertex Algebras and Relative Vertex Operators*, Progress in Math., Vol. 112, Birkhäuser, Boston, 1993.
- [Dr] V. G. Drinfeld, On quasitriangular quasi-Hopf algebras and a group closely connected with $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, *Algebra Anal.* **2** (1990), 149–181.
- [FGST1] B. L. Feigin, A. M. Gainutdinov, A. M. Semikhatov and I. Yu. Tipunin, Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center, *Comm. Math. Phys.* **265** (2006), 47–93.

- [FGST2] B. L. Feigin, A. M. Gainutdinov, A. M. Semikhatov and I. Yu. Tipunin, Kazhdan-Lusztig correspondence for the representation category of the triplet W -algebra in logarithmic CFT, to appear; math.QA/0512621.
- [Fi1] M. Finkelberg, Fusion categories, Ph.D. thesis, Harvard University, 1993.
- [Fi2] M. Finkelberg, An equivalence of fusion categories, *Geom. Funct. Anal.* **6** (1996), 249–267.
- [Fl1] M. Flohr, On modular invariant partition functions of conformal field theories with logarithmic operators, *Internat. J. Modern Phys. A* **11** (1996), 4147–4172.
- [Fl2] M. Flohr, Bits and pieces in logarithmic conformal field theory, *Internat. J. Modern Phys. A* **18** (2003), 4497–4591.
- [FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.
- [FLM1] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex operator calculus, in: *Mathematical Aspects of String Theory, Proc. 1986 Conference, San Diego*. ed. by S.-T. Yau, World Scientific, Singapore, 1987, 150–188.
- [FLM2] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.
- [FZ] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123–168.
- [Fu] J. Fuchs, On non-semisimple fusion rules and tensor categories, in: *Lie Algebras, Vertex Operator Algebras and Their Applications, a Conference in Honor of J. Lepowsky and R. Wilson*, ed. by Y.-Z. Huang and K. C. Misra, Contemp. Math., Amer. Math. Soc., to appear.
- [FHST] J. Fuchs, S. Hwang, A. M. Semikhatov and I. Yu. Tipunin, Nonsemisimple fusion algebras and the Verlinde formula, *Comm. Math. Phys.* **247** (2004), 713–742.
- [Ga1] M. Gaberdiel, Fusion rules and logarithmic representations of a WZW model at fractional level, *Nucl. Phys.* **B618** (2001), 407–436.
- [Ga2] M. Gaberdiel, An algebraic approach to logarithmic conformal field theory, *Internat. J. Modern Phys. A* **18** (2003), 4593–4638.
- [GK1] M. Gaberdiel and H. G. Kausch, A rational logarithmic conformal field theory, *Phys. Lett.* **B386** (1996), 131–137.
- [GK2] M. Gaberdiel and H. G. Kausch, A local logarithmic conformal field theory, *Nucl. Phys.* **B538** (1999), 631–658.

- [Gu] V. Gurarie, Logarithmic operators in conformal field theory, *Nucl. Phys.* **B410** (1993), 535–549.
- [He] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 2, Wiley Interscience [John Wiley & Sons], New York-London-Sydney, 1977.
- [H1] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.* 100 (1995) 173–216.
- [H2] Y.-Z. Huang, Virasoro vertex operator algebras, (nonmeromorphic) operator product expansion and the tensor product theory, *J. Alg.* **182** (1996), 201–234.
- [H3] Y.-Z. Huang, A nonmeromorphic extension of the moonshine module vertex operator algebra, in: *Moonshine, the Monster, and related topics*, South Hadley, MA, 1994, Contemp. Math., Vol. 193, Amer. Math. Soc., Providence, RI, 1996, 123–148.
- [H4] Y.-Z. Huang, *Two-dimensional Conformal Field Theory and Vertex Operator Algebras*, Progress in Math., Vol. 148, Birkhäuser, Boston, 1997.
- [H5] Y.-Z. Huang, Conformal-field-theoretic analogues of codes and lattices, in: *Kac-Moody Lie Algebras and Related Topics, Proc. Ramanujan International Symposium on Kac-Moody Lie algebras and applications*, ed. N. Sthanumoorthy and K. C. Misra, Contemp. Math., Vol. 343, Amer. Math. Soc., Providence, RI, 2004, 131–145.
- [H6] Y.-Z. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375–400.
- [H7] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, *Comm. Contemp. Math.*, to appear; math.QA/0502533.
- [HL1] Y.-Z. Huang and J. Lepowsky, Toward a theory of tensor products for representations of a vertex operator algebra, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 344–354.
- [HL2] Y.-Z. Huang and J. Lepowsky, Vertex operator algebras and operads, in: *The Gelfand Mathematical Seminars, 1990–1992*, ed. L. Corwin, I. Gelfand and J. Lepowsky, Birkhäuser, Boston, 1993, 145–161.
- [HL3] Y.-Z. Huang and J. Lepowsky, Operadic formulation of the notion of vertex operator algebra, in: *Proc. 1992 Joint Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups, Mount Holyoke, 1992*, Contemp. Math., Vol. 175, Amer. Math. Soc., Providence, RI, 1994, 131–148.

- [HL4] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebras and vertex tensor categories, in: *Lie Theory and Geometry, in honor of Bertram Kostant*, ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, V. Kac, Birkhäuser, Boston, 1994, 349–383.
- [HL5] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699–756.
- [HL6] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica (New Series)* **1** (1995), 757–786.
- [HL7] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995) 141–171.
- [HL8] Y.-Z. Huang and J. Lepowsky, Intertwining operator algebras and vertex tensor categories for affine Lie algebras, *Duke Math. J.* **99** (1999), 113–134.
- [HL9] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, V, to appear.
- [HLLZ] Y.-Z. Huang, J. Lepowsky, H. Li and L. Zhang, On the concepts of intertwining operator and tensor product module in vertex operator algebra theory, *J. Pure Appl. Algebra* **204** (2006), 507–535.
- [HLZ] Y.-Z. Huang, J. Lepowsky and L. Zhang, A logarithmic generalization of tensor product theory for modules for a vertex operator algebra, *Internat. J. Math.* **17** (2006), 975–1012.
- [HM1] Y.-Z. Huang and A. Milas, Intertwining operator superalgebras and vertex tensor categories for superconformal algebras, I, *Comm. Contemp. Math.* **4** (2002), 327–355.
- [HM2] Y.-Z. Huang and A. Milas, Intertwining operator superalgebras and vertex tensor categories for superconformal algebras, II, *Trans. Amer. Math. Soc.* **354** (2002), 363–385.
- [J] K. Jacobs, *Measure and integral*, Academic Press, New York, 1978.
- [KW] V. Kac and W. Wang, Vertex operator superalgebras and their representations, in: *Mathematical aspects of conformal and topological field theories and quantum groups*, South Hadley, MA, 1992, Contemp. Math., Vol. 175, Amer. Math. Soc., Providence, RI, 1994, 161–191.
- [KL1] D. Kazhdan and G. Lusztig, Affine Lie algebras and quantum groups, *Duke Math. J., IMRN* **2** (1991), 21–29.

- [KL2] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, I, *J. Amer. Math. Soc.* **6** (1993), 905–947.
- [KL3] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, II, *J. Amer. Math. Soc.* **6** (1993), 949–1011.
- [KL4] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, III, *J. Amer. Math. Soc.* **7** (1994), 335–381.
- [KL5] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, IV, *J. Amer. Math. Soc.* **7** (1994), 383–453.
- [KZ] V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and Wess-Zumino models in two dimensions, *Nuclear Phys. B* **247** (1984), 83–103.
- [Leo] A. F. Leont’ev, *Exponential Series* (in Russian), Nauka, Moscow, 1976.
- [LL] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Math., Vol. 227, Birkhäuser, Boston, 2003.
- [Li] H. Li, An analogue of the Hom functor and a generalized nuclear democracy theorem, *Duke Math. J.* **93** (1998), 73–114.
- [Lu] D. Lubell, Problem 10992, Problems and Solutions, *American Mathematical Monthly* **110** (2003), 155.
- [M] Y. I. Mel’nik, Uniqueness of expansion of a function, regular in a convex polygon, into a series of exponential functions, *Ukrainian Math. J.* **34** (1982), 176–178.
- [Mi] A. Milas, Weak modules and logarithmic intertwining operators for vertex operator algebras, in *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory*, ed. S. Berman, P. Fendley, Y.-Z. Huang, K. Misra, and B. Parshall, Contemp. Math., Vol. 297, American Mathematical Society, Providence, RI, 2002, 201–225.
- [MS] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.
- [NT] K. Nagatomo and A. Tsuchiya, Conformal field theories associated to regular chiral vertex operator algebras I: theories over the projective line, *Duke Math. J.* **128** (2005), 393–471.
- [RT] M. R. Rahimi Tabar, Disorder systems and logarithmic conformal field theory, *Internat. J. Modern Phys. A* **18** (2003), 4703–4745.
- [TUY] A. Tsuchiya, K. Ueno and Y. Yamada, Conformal field theory on universal family of stable curves with gauge symmetries, in: *Advanced Studies in Pure Math.*, Vol. 19, Kinokuniya Company Ltd., Tokyo, 1989, 459–566.

- [T] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Math., Vol. 18, Walter de Gruyter, Berlin, 1994.
- [W] W. Wang, Rationality of Virasoro vertex operator algebras, *Duke Math. J.* **71** (1993), 197–211.
- [Z1] L. Zhang, Ph.D. thesis, Rutgers University, 2004.
- [Z2] L. Zhang, Vertex tensor category structure on a category of Kazhdan-Lusztig, to appear.

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